1. ◇ If M^n is a connected, orientable, and compact n-manifold with $H_1(M^n; Z) = 0$ and if $N^{n-1} \subset M^n$ is a compact connected $(n-1)$-manifold, then show that $M^n - N^{n-1}$ has exactly two components with N^{n-1} as the topological boundary of each.

2. Give a counterexample to Problem 1 if the condition $H_1(M^n; Z) = 0$ is dropped.

3. Show, by example, that Corollary 8.8 would be false if $\check{H}$ were replaced by H.

4. For a locally compact space X, define $H_c^p(X) = \varinjlim H^p(X, X - K)$ where K ranges over the compact subsets of X. (This is called "cohomology with compact supports.") For an oriented n-manifold M^n, define a cap product $\cap \vartheta \colon H_c^p(M^n) \to H_{n-p}(M^n)$ and show that it is an isomorphism. (Hint: For $U \subset X$ open with $\bar{U}$ compact, $\varinjlim \check{H}^p(X, X - U) = \varinjlim H^p(X, X - \bar{U})$.)

5. Using Problem 4, show that, for a connected n-manifold M^n, $H_c^n(M^n) \approx Z$ for M^n orientable and $H_c^n(M^n) \approx Z_2$ for M^n nonorientable.

6. If M^{2n+1} is a compact connected $(2n+1)$-manifold, possibly nonorientable, show that the Euler characteristic of M^{2n+1} is zero. (Assume the fact that $H_*(M)$ is finitely generated.)

7. If M^3 is a compact, connected, and nonorientable 3-manifold, show that $H_1(M)$ is infinite. (Hint: Use Problem 6.)

8. If $U \subset R^3$ is open, show that $H_1(U)$ is torsion free. (Hint: This would be false for $U \subset R^n, n > 3$.)

9. Show that Corollary 8.9 remains true if the hypothesis that $H_1(M; Z) = 0$ is weakened to $H_1(M; Z_2) = 0$.

10. Rework Problems 6–9 of Section 19 of Chapter IV in light of the results of the present section.

9. Duality on Compact Manifolds with Boundary

We remark that, in general, if M^n is compact then the orientation ϑ is simply an element of $H_n(M^n)$ which is a generator on each component. In this case, we usually denote it by $[M] \in H_n(M)$. This class $[M]$ is called the "orientation class" or "fundamental class" of M.

Let M^n be a compact n-manifold with boundary ∂M. We shall assume that there is a neighborhood of ∂M in M^n which is a product $\partial M \times [0, 2)$, with ∂M corresponding to $\partial M \times \{0\}$. This is clearly the case for smooth manifolds and it is also known to always be the case for paracompact topological manifolds, by a theorem of M. Brown [2]. Also, one can avoid such an assumption merely by adding an external collar. For simplicity of notation, we will treat $\partial M \times [0, 2)$ as a subspace of M.

Assume that M^n is connected and orientable, by which we mean that its interior $M - \partial M$ is orientable. Then we have the following isomorphisms:

$$H_n(M, \partial M) \approx H_n(M, \partial M \times [0, 1)) \qquad \text{(homotopy)}$$

$$\approx H_n(\text{int}(M), \partial M \times (0, 1)) \quad \text{(excision)}$$

$$\approx H^0(M - \partial M \times [0, 1)) \quad \text{(duality)}$$

$$\approx H^0(M) \qquad\qquad\qquad \text{(homotopy)}$$

$$\approx \mathbf{Z}.$$

The orientation class $\vartheta \in H_n(\text{int}(M), \partial M \times (0, 1))$ corresponds to a class $[M] \in H_n(M, \partial M)$. At the other end of this sequence of isomorphisms, the orientation class corresponds to $1 \in H^0(M)$, the class of the augmentation cocycle taking all 0-simplices to 1.

Consider the following sequence of isomorphisms:

$$H^p(M; G) \approx H^p(M - \partial M \times [0, 1); G) \qquad \text{(homotopy)}$$

$$\approx H_{n-p}(\text{int}(M), \partial M \times (0, 1); G) \quad \text{(duality, cap with } \vartheta)$$

$$\approx H_{n-p}(M, \partial M \times [0, 1); G) \qquad \text{(excision)}$$

$$\approx H_{n-p}(M, \partial M; G) \qquad\qquad\quad \text{(homotopy)}.$$

By naturality of the cap product, the resulting isomorphism $H^p(M; G) \approx H_{n-p}(M, \partial M; G)$ is the cap product with the orientation class $[M] \in H^n(M, \partial M)$.

9.1. Lemma. *If M^n is compact and orientable then ∂M is orientable and $[\partial M] = \partial_*[M]$ is an orientation class, where ∂_* is the connecting homomorphism of the exact sequence of the pair $(M, \partial M)$.*

PROOF. Let A be a component of ∂M, and put $B = \partial M - A$ (possibly empty). Consider the exact homology sequence of the triple $(M, A \cup B, B)$. Part of it is the homomorphism $\partial_*: H_n(M, A \cup B) \to H_{n-1}(A \cup B, B)$. The first group is $H_n(M, \partial M)$ and the second is isomorphic, by excision, to $H_{n-1}(A)$. If c is a chain representing $[M] \in H_n(M, \partial M)$ then $[M] = [\![c]\!]$ goes to $[\![\text{part of } \partial c \text{ in } A]\!]$ in $H_{n-1}(A)$. Thus we are to show that the part of ∂c in A is an orientation, i.e., that it gives a generator of $H_{n-1}(A)$.

For any coefficient group G, we have

$$H_n(M, B; G) \approx H_n(M, B \times [0, 1); G)$$

$$\approx H_n(\text{int}(M), B \times (0,1); G)$$

$$\approx \Gamma_c(\text{int}(M) - B \times (0, 1), \Theta \otimes G)$$

$$= 0,$$

since $\text{int}(M) - B \times (0, 1)$ is connected and *non*-compact. By the Universal Coefficient Theorem,

$$0 = H_n(M, B; \mathbf{Q}/\mathbf{Z}) \approx H_n(M, B) \otimes \mathbf{Q}/\mathbf{Z} \oplus TH_{n-1}(M, B),$$

see Example 7.6 of Chapter V. Hence, $H_{n-1}(M, B)$ is torsion free and the exact sequence of the triple $(M, A \cup B, B)$ has the segment

$$0 \to H_n(M, \partial M) \xrightarrow{\partial_*} H_{n-1}(A) \to (\text{torsion free}).$$

But $H_n(M, \partial M) \approx \mathbf{Z}$, and $H_{n-1}(A)$ is either $\mathbf{Z}$ (if orientable) or 0 (if not). Thus $H_{n-1}(A)$ must be $\mathbf{Z}$ and ∂_* must be onto to make the cokernel torsion free. $\square$

9.2. Theorem. *If M^n is an oriented, compact, connected n-manifold with boundary, then the diagram (arbitrary coefficients)*

$$\cdots \longrightarrow H^p(M) \overset{i^*}{\longrightarrow} H^p(\partial M) \overset{\delta^*}{\longrightarrow} H^{p+1}(M, \partial M) \overset{j^*}{\longrightarrow} H^{p+1}(M) \longrightarrow \cdots$$
$$\approx \Big\downarrow \cap [M] \quad (-1)^p \quad \approx \Big\downarrow \cap [\partial M] \quad (-1)^{p+1} \quad \approx \Big\downarrow \cap [M] \quad 1 \quad \approx \Big\downarrow \cap [M]$$
$$\cdots \longrightarrow H_{n-p}(M, \partial M) \overset{}{\underset{\partial_*}{\longrightarrow}} H_{n-p-1}(\partial M) \overset{}{\underset{i_*}{\longrightarrow}} H_{n-p-1}(M) \overset{}{\underset{j_*}{\longrightarrow}} H_{n-p-1}(M, \partial M) \longrightarrow \cdots$$

with exact rows, commutes up to the indicated signs. This also holds, without the orientability restriction, over the base ring $\mathbf{Z}_2$.

PROOF. All the vertical isomorphisms, except the third, result from previous theorems or remarks. The third one will follow from the 5-lemma as soon as we have proved the commutativity.

Let $c \in \Delta_n(M)$ represent the orientation class $[M] \in H_n(M, \partial M)$. Then ∂c is a chain in ∂M.

For the first square, let $f \in \Delta^p(M)$ be a cocycle. Then going right and then down gives a class represented by $f|_{\partial M} \cap \partial c = f \cap \partial c = (-1)^p \partial(f \cap c)$. Going down then right gives $\partial(f \cap c)$.

For the second square, let $f \in \Delta^p(M)$ with $\delta f = 0$ on ∂M. Then going right then down gives $(\delta f) \cap c = \partial(f \cap c) + (-1)^{p+1} f \cap \partial c$ which is homologous to $(-1)^{p+1} f \cap \partial c = (-1)^{p+1} f|_{\partial M} \cap \partial c$. Going down then right gives $f|_{\partial M} \cap \partial c$.

Commutativity of the third square is obvious. $\square$

9.3. Corollary. $\cap [M]: H^p(M, \partial M; G) \to H_{n-p}(M; G)$ *is an isomorphism.* $\square$

It is often desirable to have a version of duality entirely in terms of cohomology and the cup product. To this end, let Λ be a principal ideal domain and, with the notation of Example 7.6 of Chapter V, put

$$\bar{H}^p(\cdot) = H^p(\cdot)/TH^p(\cdot),$$

the "torsion free part" of the pth cohomology group. Note that if Λ is a field then $\bar{H} = H$. We shall assume the fact, proved in Appendix E, that $H_*(M; \Lambda)$ is finitely generated. Then it follows that $\text{Ext}(H_*(M), \Lambda)$ is all torsion so that the Universal Coefficient Theorem gives the isomorphism $\bar{H}^p(M; \Lambda) \overset{\approx}{\longrightarrow} \text{Hom}(\bar{H}_p(M), \Lambda)$.

9.4. Theorem. *Let M^n be a compact, connected, oriented (over Λ) n-manifold with boundary. Then the cup product pairing*

$$\bar{H}^p(M; \Lambda) \otimes_\Lambda \bar{H}^{n-p}(M, \partial M; \Lambda) \to H^n(M, \partial M; \Lambda) \approx H_0(M; \Lambda) \approx \Lambda$$

taking $(\alpha \otimes \beta) \mapsto \alpha \cup \beta \mapsto \langle \alpha \cup \beta, [M] \rangle \in \Lambda$, is a duality pairing. That is, the map

$$\bar{H}^p(M; \Lambda) \to \mathrm{Hom}_\Lambda(\bar{H}^{n-p}(M, \partial M; \Lambda), \Lambda),$$

taking $\alpha \mapsto \bar{\alpha}$ where $\bar{\alpha}(\beta) = \langle \alpha \cup \beta, [M] \rangle$, is an isomorphism.

PROOF. We have the isomorphism $\bar{H}^p(M; \Lambda) \xrightarrow{\approx} \mathrm{Hom}(\bar{H}_p(M), \Lambda) \xrightarrow{\approx}$ $\mathrm{Hom}(\bar{H}^{n-p}(M, \partial M), \Lambda)$ (given by the Universal Coefficient Theorem and cap with $[M]$, respectively), taking, say, α to α^* and then to α°, where $\alpha^*(\gamma) = \langle \alpha, \gamma \rangle$. We claim that $\alpha^\circ = \bar{\alpha}$, which would prove the desired isomorphism. We compute $\alpha^\circ(\beta) = \alpha^*(\beta \cap [M]) = \langle \alpha, \beta \cap [M] \rangle = \langle \alpha \cup \beta, [M] \rangle = \bar{\alpha}(\beta)$. $\square$

PROBLEMS

1. If M^n and N^n are compact connected oriented n-manifolds, one defines their "connected sum" $M \# N$ as follows: Take a nicely embedded n-disk in each, remove its interior, and paste the remainders together via an orientation *reversing* homeomorphism on the boundary spheres of these disks. Show that the cohomology ring of $M \# N$ is isomorphic to the ring resulting from the direct product of the rings for M and N with the unity elements (in dimension 0) identified and the orientation classes identified. Similarly, the multiples of these identifications must also be made. (The orientation cohomology class of M is that class $\vartheta \in H^n(M)$ which is Kronecker dual to $[M]$, i.e., such that $\vartheta[M] = 1$. It can also be described as the class that is Poincaré dual to the standard generator in $H_0(M)$, the class represented by any 0-simplex.) In particular, cup products of positive dimensional classes, one from each of the two original manifolds, are zero.

2. Suppose that N^n is a compact, orientable, smooth n-manifold embedded smoothly in the compact, orientable m-manifold M^m. Let W be a closed tubular neighborhood of N in M. Show that there exists an isomorphism $H_p(N) \approx H_{m-n+p}(W, \partial W)$.

3. ✧ Let M^n be a compact manifold with boundary $\partial M = A \cup B$ where A and B are $(n-1)$-manifolds with common boundary $A \cap B$. Since $A \cap B$ is a neighborhood retract in both A and B (see Appendix E) the inclusion $\Delta_*(A) + \Delta_*(B) \hookrightarrow \Delta_*(A \cup B)$ induces an isomorphism in homology, and so there is a cap product

$$\cap : H^p(M, A) \otimes H_n(M, A \cup B) \to H_{n-p}(M, B).$$

Take the orientation class $[A]$ to come from $[\partial M] = \partial_*[M]$ via $H_{n-1}(\partial M) = H_{n-1}(A \cup B) \to H_{n-1}(A \cup B, B) \approx H_{n-1}(A, A \cap B)$ (by excision and homotopy). Show that the diagram

commutes up to sign. Deduce that there is the duality isomorphism

$$\boxed{\cap [M] : H^p(M, A) \xrightarrow{\approx} H_{n-p}(M, B).}$$

4. Verify, by direct computation, the isomorphism $H^p(M, A) \approx H_{3-p}(M, B)$ for $M^3 = S^1 \times D^2$ and where A is a nice 2-disk in ∂M and B is the closure of the complement of A in ∂M.

5. If M^m and N^n are compact orientable manifolds with boundary, show that $H_{m-p}((M, \partial M) \times N) \approx H^{n+p}(M \times (N, \partial N))$.

10. Applications of Duality

In this section we will give several applications of duality to problems about manifolds. It is standard terminology to refer to compact manifolds without boundary as "closed" manifolds. We shall occasionally use the fact, from Appendix E, that such manifolds have finitely generated homology.

10.1. Proposition. *Let M^n be a closed, connected, orientable manifold and let $f: S^n \to M$ be a map of nonzero degree. Then $H_*(M^n; \mathbf{Q}) \approx H_*(S^n; \mathbf{Q})$. If, moreover, $\deg(f) = \pm 1$, then $H_*(M^n; \mathbf{Z}) \approx H_*(S^n; \mathbf{Z})$.*

PROOF. For the last part, suppose $H_q(M; \mathbf{Z}) \neq 0$ for some $q \neq 0, n$. Then it can easily be seen from the Universal Coefficient Theorem that there is a field Λ such that $H^q(M; \Lambda) \neq 0$. For the first part, take $\Lambda = \mathbf{Q}$.

If $0 \neq \alpha \in H^q(M; \Lambda)$ then there is a $\beta \in H^{n-q}(M; \Lambda)$ with $\alpha \cup \beta \neq 0$. Thus $\alpha\beta = k \cdot \gamma$, where γ is a generator of $H^n(M; \Lambda)$ and $0 \neq k \in \Lambda$. Therefore $0 = 0 \cdot 0 = f^*(\alpha) f^*(\beta) = f^*(\alpha\beta) = f^*(k\gamma) = k \cdot \deg(f) \cdot \text{generator} \neq 0$. $\square$

10.2. Proposition. *The cohomology rings of the real, complex, and quaternionic projective spaces are:*

$$H^*(\mathbf{RP}^n; \mathbf{Z}_2) \approx \mathbf{Z}_2[\alpha]/\alpha^{n+1} \qquad \textit{where} \quad \deg(\alpha) = 1,$$
$$H^*(\mathbf{CP}^n; \mathbf{Z}) \approx \mathbf{Z}[\alpha]/\alpha^{n+1} \qquad \textit{where} \quad \deg(\alpha) = 2,$$
$$H^*(\mathbf{QP}^n; \mathbf{Z}) \approx \mathbf{Z}[\alpha]/\alpha^{n+1} \qquad \textit{where} \quad \deg(\alpha) = 4.$$

PROOF. We already know the additive (co)homology groups of these spaces. The arguments for all of these are essentially the same so we will give it only for the case of complex projective space. The proof is by induction on n. Suppose it holds for $n - 1$, i.e., that there is an element $\alpha \in H^2(\mathbf{CP}^{n-1})$ such that $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ generate the homology groups in those dimensions. Now $\mathbf{CP}^n$ is obtained from $\mathbf{CP}^{n-1}$ by attaching a $2n$-cell. It follows from the exact sequence of the pair $(\mathbf{CP}^n, \mathbf{CP}^{n-1})$ that $H^i(\mathbf{CP}^n) \to H^i(\mathbf{CP}^{n-1})$ is an isomorphism for $i \leq 2n - 2$. Thus it makes sense to identify α and its powers up to α^{n-1} with their preimages in $H^i(\mathbf{CP}^n)$ in this range. (This is just a notational convenience.) Also, of course, the case $n = 1$ is trivial, so we can assume $n \geq 2$. Thus we have the classes $\alpha \in H^2(\mathbf{CP}^n)$ and $\alpha^{n-1} \in H^{2n-2}(\mathbf{CP}^n)$. By Theorem 9.4, the product $\alpha^n = \alpha \cup \alpha^{n-1}$ must be a generator of $H^{2n}(\mathbf{CP}^n)$. $\square$

10.3. Corollary. *Any homotopy equivalence* $\mathbf{CP}^{2n} \to \mathbf{CP}^{2n}$ *preserves orientation for* $n \geq 1$.

PROOF. Such a map f must be an isomorphism on $H^2(\mathbf{CP}^{2n}) \approx \mathbf{Z}$ and so, for a generator α we must have $f^*(\alpha) = \pm \alpha$. Therefore $f^*(\alpha^{2n}) = (f^*(\alpha))^{2n} = (\pm \alpha)^{2n} = \alpha^{2n}$. The contention follows since this is a top dimensional generator. $\square$

We will now study to a small extent the cohomology of manifolds that are boundaries of other manifolds.

10.4. Theorem. *Let* Λ *be a field* (*coefficients for all homology and cohomology*). *Let* V^{2n+1} *be an oriented* (*unless* $\Lambda = \mathbf{Z}_2$) *compact manifold with* $\partial V = M^{2n}$ *connected. Then* $\dim H^n(M^{2n})$ *is even and*

$$\dim[\ker(i_*\colon H_n(M) \to H_n(V))] = \dim[\operatorname{im}(i^*\colon H^n(V) \to H^n(M))] = \tfrac{1}{2}\dim H^n(M).$$

Moreover, $\operatorname{im}(i^*) \subset H^n(M)$ *is self-annihilating, i.e., the cup product of any two classes in it is zero.*

PROOF. Consider this portion of the Poincaré–Lefschetz diagram:

$$
\begin{array}{ccccc}
H^n(V) & \xrightarrow{\;\;i^*\;\;} & H^n(M) & \xrightarrow{\;\;\delta^*\;\;} & H^{n+1}(V, M) \\[2pt]
& & \cap[M]\Big\downarrow \approx & & \approx \Big\downarrow \cap[V] \\[2pt]
& & H_n(M) & \xrightarrow{\;\;i_*\;\;} & H_n(V).
\end{array}
$$

From the diagram we see that $\{\operatorname{im}(i^*)\} \cap [M] = \{\ker(\delta^*)\} \cap [M] = \ker(i_*)$. Thus $\operatorname{rank}(i^*) = \dim \operatorname{im}(i^*) = \dim \ker(i_*) = \dim H_n(M) - \operatorname{rank}(i_*) = \dim H^n(M) - \operatorname{rank}(i^*)$, since i^* and i_* are Kronecker duals of one another (this is the fact that the rank of a transposed matrix equals the rank of the original). Therefore, $\dim H^n(M) = 2 \cdot \operatorname{rank}(i^*) = 2 \cdot \dim(\ker(i_*))$.

Now if $\alpha, \beta \in H^n(V)$ then $\delta^*(i^*(\alpha) \cup i^*(\beta)) = (\delta^* i^*)(\alpha \cup \beta) = 0$ since $\delta^* i^* = 0$ by exactness. But $\delta^*\colon H^{2n}(M) \to H^{2n+1}(V, M)$ is a monomorphism since it is dual to $i_*\colon H_0(M) \to H_0(V)$. Thus $i^*(\alpha) \cup i^*(\beta) = 0$ as claimed. $\square$

10.5. Corollary. *If* $M^m = \partial V$ *is connected with* V *compact, then the Euler characteristic* $\chi(M)$ *is even; also see Problem 1.*

PROOF. If $\dim(M)$ is odd then Poincaré duality on M pairs odd and even dimensions and so $\chi(M) = 0$ for all closed M. For M of dimension $2n$, we have that $\chi(M) \equiv \dim H^n(M; \mathbf{Z}_2)$ modulo 2. For $M = \partial V$, the latter is 0 (mod 2) by Theorem 10.4. $\square$

10.6. Corollary. $\mathbf{RP}^{2n}, \mathbf{CP}^{2n}$, *and* $\mathbf{QP}^{2n}$ *are not boundaries of compact manifolds.* $\square$

We remark that all *orientable* two- and three-dimensional closed manifolds are boundaries. The Klein bottle is a nonorientable 2-manifold which is a boundary.

10.7. Definition (H. Weyl). Let M be a closed oriented manifold. The *signature* of M is defined to be 0 if dim(M) is not divisible by 4. If dim(M) = $4n$, then signature(M) is defined to be the signature of the quadratic form $\langle \alpha, \beta \rangle = (\alpha \cup \beta)[M]$ on $H^{2n}(M; \mathbf{R})$.

Recall that a quadratic form over the reals is the sum and difference of squares. Its "signature" is the sum of the signs on those squares. Another term used for this is "index."

10.8. Corollary (Thom). *If* $M^{4n} = \partial V^{4n+1}$ *is connected with* V *compact and orientable then* signature(M) $= 0$.

PROOF. Let $W = H^{2n}(M; \mathbf{R})$ and let dim(W) $= 2k$. The quadratic form (over $\mathbf{R}$) of Definition 10.7 is equivalent to the sum of, say, r positive squares and, thus, $2k - r$ negative ones. That is, there is a subspace W^+ on which the form is positive definite and another subspace W^- on which it is negative definite with dim $W^+ = r$ and dim $W^- = 2k - r$. By Theorem 10.4, there is a subspace $U \subset W$ of dimension k such that $\langle \alpha, \beta \rangle = 0$ on U. Clearly $U \cup W^+ = \{0\}$ and so the sum $r + k$ of their dimensions cannot be greater than the dimension $2k$ of W. That is, $r + k \leq 2k$, so that $r \leq k$.

Similarly $U \cap W^- = \{0\}$, so that $(2k - r) + k \leq 2k$, i.e., $k \leq r$.

Thus $r = k$ and the signature is zero. $\qquad\square$

10.9. Example. The connected sum (see Problem 1 in Section 9) $M^4 = \mathbf{CP}^2 \# \mathbf{CP}^2$ is not the boundary of an orientable 5-manifold. To see this, note that the ring of M^4 is generated by classes $\alpha, \beta \in H^2(M)$, with $\alpha\beta = 0$ and $\alpha^2 = \beta^2$, so that its quadratic form is the identity 2×2 matrix whose signature is 2 (or -2 for the other orientation).

Of course, a more general argument shows that the signature is additive with respect to the connected sum operation on oriented manifolds.

However, $\mathbf{CP}^2 \# - \mathbf{CP}^2$ is the boundary of the orientable 5-manifold $V^5 = (\mathbf{CP}^2 - U) \times I$, where U is an open 4-disk in $\mathbf{CP}^2$. ($-\mathbf{CP}^2$ stands for $\mathbf{CP}^2$ with the opposite orientation.) The only difference in the cohomology ring is that $\beta^2 = -\alpha^2$, but that is enough, of course, to make the signature zero. Naturally, this is a general fact having nothing to do with $\mathbf{CP}^2$ specifically.

Also we claim that $\mathbf{CP}^2 \# \mathbf{CP}^2$ is the boundary of a nonorientable 5-manifold. To see this consider $(\mathbf{CP}^2 \times I) \# (\mathbf{RP}^2 \times S^3)$, where the sum is done away from the two boundary components. Now run an arc from one of the boundary components through an orientation reversing loop in $\mathbf{RP}^2 \times S^3$ and then to the other boundary component. Done nicely this arc has a product neighborhood, and we can remove that. This leaves $\mathbf{CP}^2 \# \mathbf{CP}^2$ as the