

It follows that  $\pi_n(E)$  and  $\pi_{n-1}(E)$  are both trivial. The remainder follows from the rest of the top sequence.  $\square$

**11.11. Definition.** A map  $f: X \rightarrow Y$  is an  $n$ -equivalence if  $f_\#: \pi_i(X) \rightarrow \pi_i(Y)$  is an isomorphism for  $i < n$  and an epimorphism for  $i = n$ . If  $f$  is an  $n$ -equivalence for all  $n$  then it is called a *weak homotopy equivalence* or an  $\infty$ -equivalence.

Note that the condition in Definition 11.11 is equivalent to  $\pi_i(M_f, X)$  being 0 for  $i \leq n$ .

**11.12. Theorem.** For  $n \leq \infty$ , a map  $f: X \rightarrow Y$  is an  $n$ -equivalence if and only if, for every relative CW-pair  $(K, L)$  with  $\dim(K - L) \leq n$  any commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{h} & X \\ \downarrow i & & \downarrow f \\ K & \xrightarrow{g} & Y \end{array}$$

can be completed to

$$\begin{array}{ccc} L & \xrightarrow{h} & X \\ \downarrow i & \searrow \simeq & \downarrow f \\ K & \xrightarrow{g} & Y \end{array}$$

where the top triangle commutes and the bottom triangle commutes up to a homotopy rel  $L$ .

**PROOF.** The implication  $\Leftarrow$  is just the definition of  $\pi_i(M_f, X) = 0$  using  $(K, L) = (\mathbf{D}^i, \mathbf{S}^{i-1})$ .

Let  $i: X \hookrightarrow M_f$  be the inclusion and  $p: M_f \rightarrow Y$  the projection. Since  $M_f \supset Y$ , the map  $g$  can be regarded as a map  $g: K \rightarrow M_f$  and  $f$  can be regarded as a map  $f: X \rightarrow M_f$ . Then  $g|_L = f \circ h \simeq i \circ h$  since  $f \simeq i: X \rightarrow M_f$ . By the homotopy extension property applied to  $(K, L)$ , there is a homotopy  $F: K \times I \rightarrow M_f$  of  $g$  to a map  $g': K \rightarrow M_f$  such that  $g'|_L = i \circ h$ , and  $p \circ F: K \times I \rightarrow Y$  is a homotopy rel  $L$ . Thus  $p \circ g \simeq p \circ g' \text{ rel } L$ .

Now extend the map  $h \times I \cup g' \times \{0\}: L \times I \cup K \times \{0\} \rightarrow M_f$  to  $G: K \times I \rightarrow M_f$  such that  $G(K \times \{1\}) \subset X$  by induction over skeletons of  $(K, L)$  using that  $\pi_i(M_f, X) = 0$  for  $i \leq n$  and that  $\dim(K - L) \leq n$ . Define  $\phi: K \rightarrow X$  by  $\phi(x) = G(x, 1) \in X$ .

Then for  $x \in L$ ,  $\phi(x) = G(x, 1) = G(x, 0) = h(x)$ , meaning that the top triangle commutes. Also,  $i \circ \phi = G(\cdot, 1) \simeq G(\cdot, 0) = g' \text{ rel } L$ . Thus  $f \circ \phi = p \circ i \circ \phi \simeq p \circ g' \simeq p \circ g = g \text{ rel } L$ .  $\square$

**11.13. Corollary.** If  $f: X \rightarrow Y$  is an  $n$ -equivalence ( $n \leq \infty$ ) and  $K$  is a CW-

complex, then

$$f_{\#}: [K; X] \rightarrow [K; Y]$$

is bijective for  $\dim(K) < n$  and surjective for  $\dim(K) = n$ . This also holds in the pointed category.

PROOF. The onto part is by application of Theorem 11.12 to  $(K, \emptyset)$ . The one-one part is by application of Theorem 11.12 to  $(K \times I, K \times \partial I)$ . In the pointed category use the base point instead of  $\emptyset$ .  $\square$

**11.14. Corollary.** *Let  $f: K \rightarrow L$  be a map between connected CW-complexes. Then  $f$  is a homotopy equivalence if and only if  $f_{\#}: \pi_i(K) \rightarrow \pi_i(L)$  is an isomorphism for all  $i$ .*

PROOF. Select base points corresponding under  $f$  and restrict attention to pointed maps. Then  $f_{\#}: [L; K] \rightarrow [L; L]$  is bijective by Corollary 11.13. Thus there is a  $[g] \in [L; K]$  with  $f_{\#}[g] = [1]$ . But  $f_{\#}[g] = [f \circ g]$ , so  $f \circ g \simeq 1$ .

On homotopy groups we have  $1_{\#} = (f \circ g)_{\#} = f_{\#} \circ g_{\#}$ . But  $f_{\#}$  is an isomorphism so it follows that  $g_{\#}$  is also an isomorphism in all dimensions. Then by the same argument used for  $f$  applied to  $g$ , there is a map  $h: K \rightarrow L$  such that  $g \circ h \simeq 1$ . Thus  $f \simeq f \circ g \circ h \simeq h$ , from which we get  $1 \simeq g \circ h \simeq g \circ f$ .  $\square$

**11.15. Corollary.** *Suppose that  $K$  and  $L$  are simply connected CW-complexes. If  $f: K \rightarrow L$  is such that  $f_{*}: H_i(K) \rightarrow H_i(L)$  is an isomorphism for all  $i$ , then  $f$  is a homotopy equivalence.*  $\square$

**11.16. Example.** Consider the suspension  $\Sigma(S^n \times S^m)$  of the product of two spheres,  $n, m > 0$ . We have the composition

$$\Sigma(S^n \times S^m) \rightarrow \Sigma(S^n \times S^m) \vee \Sigma(S^n \times S^m) \vee \Sigma(S^n \times S^m) \rightarrow S^{n+1} \vee S^{m+1} \vee S^{n+m+1},$$

where the first map is from the coproduct and the second is the one-point union of the maps  $\Sigma\pi_1$ ,  $\Sigma\pi_2$ , and  $\Sigma\eta$ , where  $\pi_1$  and  $\pi_2$  are the projections to the factors of the product and  $\eta: S^n \times S^m \rightarrow S^n \wedge S^m \approx S^{n+m}$ . It is easily seen that this composition is an isomorphism in homology. Thus it is a homotopy equivalence by Corollary 11.15.

**11.17. Example.** We shall prove the converse of Theorem 10.14 of Chapter VI, thereby giving a complete homotopy classification of the lens spaces  $L(p, q)$ . We must show that if  $\pm qq'$  is a quadratic residue mod  $p$  then  $L(p, q) \simeq L(p, q')$ . The condition is equivalent to the existence of integers  $k, n$ , and  $m$ , prime to  $p$ , such that  $n^2 k q' + mp = \pm 1$  and  $kq \equiv 1 \pmod{p}$ . With the notation from the proof of Lemma 10.13 of Chapter VI consider the map  $\theta: S^3 \rightarrow S^3$  given by  $\theta(u, v) = (u^{(n)}, v^{(kq'n)})$ . Then it can be checked immediately that  $\theta T_q = T_{q'} \theta$ ; i.e.,  $\theta$  carries the  $\mathbb{Z}_p$ -action generated by  $T_q$  to that generated by  $T_{q'}$ . Now consider  $p$  disjoint disks in  $S^3$  permuted by  $T_q$ . By pinching the boundaries

of these disks to points, we get a space  $W = S_0^3 \cup S_1^3 \cup \cdots \cup S_p^3$  (one point unions but at different points) and an equivariant map  $S^3 \rightarrow W$  where  $S^3$  and  $S_0^3$  have the  $T_q$ -action and the other  $S_i^3$  are permuted by  $T_q$ . Map  $W \rightarrow S^3$  by putting  $\theta$  on  $S_0^3$ , and a map of degree  $m$  on  $S_1^3$  propagated to maps of degree  $m$  on the other  $S_i^3$  by equivariance. Then the composition  $\Phi: S^3 \rightarrow W \rightarrow S^3$  has degree  $\deg(\theta) + mp = n^2 k q' + mp = \pm 1$  and carries the  $T_q$  action to the  $T_q^n$  action. Since  $\Phi$  has degree  $\pm 1$  it induces isomorphisms  $\Phi: \pi_i(S^3) \rightarrow \pi_i(S^3)$  for all  $i$ . The induced map  $\Psi: L(p, q) \rightarrow L(p, q')$  on the orbit spaces then gives isomorphisms  $\Psi_\#: \pi_i(L(p, q)) \rightarrow \pi_i(L(p, q'))$  for all  $i$ ; (see the proof of Lemma 10.13 of Chapter VI). Thus  $\Psi$  is a homotopy equivalence by Corollary 11.14 as desired. This discussion generalizes easily to prove the converse of Problem 3 of Section 10 of Chapter VI; i.e., the higher-dimensional analog of the present example.

We finish this section with a brief discussion of the classification problem in topology. This is the problem of finding a way to tell whether or not two spaces are homeomorphic. This is too ambitious, so let us modify it so as to be less demanding. Let us ask for a decision procedure to determine whether or not two finite polyhedra are homotopy equivalent. Perhaps this does not sound too ambitious, but, in fact, it is, as we now explain. Suppose we are given a group  $G$  in terms of a finite number of generators and relations. Then we can construct a finite simplicial complex having  $G$  as its fundamental group by taking a one-point union of circles, one for each generator, and then attaching 2-cells (which can be done simplicially) to kill the relations. (Perhaps such a construction should be called “fratricide.”) If we had such a decision procedure, then that procedure could be used to decide whether or not  $G$  is the trivial group (i.e., whether or not the space is simply connected). The problem of finding a decision procedure for determining whether or not a group  $G$ , defined by generators and relations, is trivial, is essentially what is known as the “word problem” in group theory. The word problem is known to be unsolvable (proved in 1955 by Novikov [1]), i.e., it is known that there exists no such decision procedure. Thus we have the following fact.

**11.18. Theorem.** *There does not exist any decision procedure for determining whether or not a given two-dimensional finite polyhedron is simply connected.*

□

Also, it follows from Section 9 of Chapter III, Problem 13 that there is no decision procedure for deciding whether or not a given 4-manifold is simply connected.

This should not be taken as discouraging. After all, the simply connected spaces make up a large segment of interest in topology. Moreover, the result can be viewed as proof that topologists will never find themselves out of work.

## PROBLEMS

1. Show that  $\pi_{n+m-1}(S^n \vee S^m) \rightarrow \pi_{n+m-1}(S^n \times S^m)$  is not an isomorphism.
2. Finish Example 11.16 by showing that the indicated map is an isomorphism in homology. (*Hint*: Show the second map is onto in homology.)
3. If  $K$  is a simply connected CW-complex with  $H_n(K) \approx \mathbb{Z}$  and  $\tilde{H}_i(K) = 0$  for  $i \neq n$ , then show that  $K \simeq S^n$ .
4. Prove this amendment to the Absolute Hurewicz Theorem: Suppose that  $X$  is  $(n-1)$ -connected,  $n \geq 2$ . Then the Hurewicz homomorphism  $h_i: \pi_i(X) \rightarrow H_i(X)$  is an isomorphism for  $i \leq n$  and an epimorphism for  $i = n+1$ . (*Hint*: Consider the pair  $(Y, X)$  where  $Y$  is a space obtained from  $X$  by attaching  $n$ -cells to kill  $\pi_n(X)$ .)
5. Consider  $\alpha \in \pi_1(S^1 \vee S^2)$  and  $\beta \in \pi_2(S^1 \vee S^2)$  given by the inclusions of the factors. Let  $f: S^2 \rightarrow S^1 \vee S^2$  represent  $2\beta - \alpha(\beta) \in \pi_2(S^1 \vee S^2)$  and put  $X = (S^1 \vee S^2) \cup_f D^3$ . Show that the inclusion  $S^1 \hookrightarrow X$  induces an isomorphism on  $\pi_1$  and on  $H_*$  but is not a homotopy equivalence.
6. A “graph” is a CW-complex of dimension 1. A “tree” is a connected graph with no cycles in the sense of graph theory; i.e., having no simple closed curves.
  - (a) Show that a tree is contractible; i.e., prove the infinite case of Lemma 7.7 of Chapter III.
  - (b) Show that a connected graph has the homotopy type of the one-point union of circles (possibly infinite in number); i.e., of a graph with a single vertex (the infinite case of Lemma 7.13 of Chapter III).
  - (c) Show that the fundamental group of any connected graph is free; i.e., prove the infinite case Theorem 7.14 of Chapter III.
  - (d) Show that a subgroup of a free group is free; i.e., prove the infinite case of Corollary 8.2 of Chapter III.
7. For any space  $X$  construct a CW-complex  $K$  and a map  $f: K \rightarrow X$  which is a weak homotopy equivalence. (This is called a “CW-approximation” to  $X$ .) Use this to remove the hypothesis in Theorem 11.7 that  $X$  is semilocally 1-connected.

## 12. Eilenberg–Mac Lane Spaces

An arcwise connected space  $Y$  is called an “Eilenberg–Mac Lane space of type  $(\pi, n)$ ” if  $\pi_n(Y) \approx \pi$  and  $\pi_i(Y) = 0$  for  $i \neq n$ . We have already met these spaces in the last section where Corollary 11.9 proved their existence as CW-complexes, where, of course,  $\pi$  must be abelian for  $n > 1$ . In this section we shall also require  $\pi$  to be abelian for  $n = 1$ . Such a space is also called simply a “space of type  $(\pi, n)$ ” or a “ $K(\pi, n)$ .”

The purpose of this section is to show that there exists a natural equivalence of functors

$$[K; K(\pi, n)] \approx H^n(K; \pi),$$

on the category of CW-complexes  $K$  and maps. (Compare Hopf’s Theorem 11.6 of Chapter V.)

Note that if  $Y$  is of type  $(\pi, n+1)$  then the loop space  $\Omega Y$  is of type  $(\pi, n)$ , as follows from the exact homotopy sequence of the path-loop fibration of  $Y$ .

Also  $[K; \Omega Y] \approx [SK; Y]$  by Lemma 4.2. It is also clear that any map  $K^{(n+1)} \rightarrow \Omega Y$  extends to  $K$  since  $\pi_i(\Omega Y) = 0$  for  $i > n$ , and any partial homotopy  $K^{(n+1)} \times I \cup K \times \partial I \rightarrow \Omega Y$  extends to  $K \times I$  for the same reason. Therefore  $[K; \Omega Y] \approx [K^{(n+1)}; \Omega Y]$ .

The sequence

$$K^{(n)} \rightarrow K^{(n+1)} \rightarrow K^{(n+1)}/K^{(n)} \rightarrow SK^{(n)} \rightarrow SK^{(n+1)} \rightarrow \dots$$

is coexact by Corollaries 1.4, 5.3, and 5.5. Thus, for  $Y$  of type  $(\pi, n+1)$ , there is the diagram

$$\begin{array}{ccccccc}
 & & [S^2 K^{(n-1)}/S^2 K^{(n-2)}; Y] & \longrightarrow & [S^2 K^{(n-1)}; Y] & \longrightarrow & 0 \\
 & & \searrow \delta_{n-1} & & \downarrow & & \\
 & & & & [SK^{(n)}/SK^{(n-1)}; Y] & & \\
 & & & & \downarrow & \searrow \delta_n & \\
 (*) & 0 \longrightarrow & [SK^{(n+1)}; Y] & \longrightarrow & [SK^{(n)}; Y] & \longrightarrow & [K^{(n+1)}/K^{(n)}; Y] \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

in which the long rows and column are exact and the diagonal maps marked  $\delta$  are defined by commutativity. The 0 at the left end of the third tier is by  $[SK^{(n+1)}/SK^{(n)}; Y] = 0$  since  $SK^{(n+1)}/SK^{(n)}$  is a bouquet of  $(n+2)$ -spheres and  $\pi_{n+2}(Y) = 0$ . Similarly, the 0 at the end of the exact column is by  $[SK^{(n-1)}; Y] = 0$ , by Corollary 11.13, since  $\dim(SK^{(n-1)}) \leq n$  and  $\pi_i(Y) = 0$  for  $i \leq n$ . The 0 on the right of the first row is for the same type of reason.

An easy diagram chase gives

$$[K; \Omega Y] \approx [SK; Y] \approx [SK^{(n+1)}; Y] \approx \ker(\delta_n)/\text{im}(\delta_{n-1}).$$

It remains to identify the maps  $\delta_n$  and  $\delta_{n-1}$ . They differ only by a change of the index  $n$ , so it suffices to look at  $\delta_n$ . This is induced by the composition

$$K^{(n+1)}/K^{(n)} \xleftarrow{\approx} K^{(n+1)} \cup CK^{(n)} \rightarrow SK^{(n)} \rightarrow SK^{(n)}/SK^{(n-1)}.$$

Recall that the first map is the homotopy equivalence given by collapsing the cone to a point. The second map is the collapse of  $K^{(n+1)}$ , and the last is the collapse of  $SK^{(n-1)}$ .

Now  $K^{(n+1)}/K^{(n)}$  is a bouquet of  $(n+1)$ -spheres, one for each  $(n+1)$ -cell  $\sigma$  of  $K$ . Similarly,  $SK^{(n)}/SK^{(n-1)} = S(K^{(n)}/K^{(n-1)})$  is a bouquet of  $(n+1)$ -spheres, one for each  $n$ -cell  $\tau$  of  $K$ .

For an  $(n+1)$ -cell  $\sigma$ , consider the characteristic map

$$f_\sigma: \mathbf{D}^{n+1} \rightarrow K^{(n+1)}.$$

This extends to

$$f_\sigma \cup Cf_{\partial\sigma}: \mathbf{D}^{n+1} \cup CS^n \rightarrow K^{(n+1)} \cup CK^{(n)}.$$

Letting  $\bar{f}_\sigma: S^{n+1} \rightarrow K^{(n+1)}/K^{(n)}$  be the (inclusion) map induced by  $f_\sigma$ , we have the commutative diagram

$$\begin{array}{ccccccc} S^{n+1} & \xleftarrow{\cong} & \mathbf{D}^{n+1} \cup CS^n & \xrightarrow{\cong} & SS^n & & S^{n+1} \\ \downarrow \bar{f}_\sigma & & \downarrow f_\sigma \cup Cf_{\partial\sigma} & & \downarrow Sf_{\partial\sigma} & & \uparrow S\bar{p}_\tau \\ K^{(n+1)}/K^{(n)} & \xleftarrow{\cong} & K^{(n+1)} \cup CK^{(n)} & \longrightarrow & SK^{(n)} & \longrightarrow & SK^{(n)}/SK^{(n-1)} = \bigvee_\tau S^{n+1} \end{array}$$

where  $\bar{p}_\tau$  is the projection of  $K^{(n)}/K^{(n-1)}$  to the  $\tau$ th sphere in the bouquet. It follows that  $\delta_n$  takes the  $\sigma$ th sphere to the  $\tau$ th sphere by the map  $S(p_\tau f_{\partial\sigma})$ ; i.e., a map of degree  $\deg(p_\tau f_{\partial\sigma})$ .

Now an element of  $[\bigvee_\sigma S^{n+1}; Y]$  can be regarded as a function that assigns to each  $(n+1)$ -cell  $\sigma$  of  $K$ , an element of  $[S^{n+1}; Y] = \pi_{n+1}(Y) = \pi$ . That is, it is a cellular cochain in  $C^{n+1}(K; \pi) = \text{Hom}(C_{n+1}(K), \pi)$ . Similarly, an element of  $[\bigvee_\tau S^{n+1}; Y]$  is a function assigning to each  $n$ -cell  $\tau$  of  $K$ , an element of  $\pi$ . That is, it is a cochain in  $C^n(K; \pi)$ . We have shown that the map  $[SK^{(n)}/SK^{(n-1)}; Y] \rightarrow [K^{(n+1)}/K^{(n)}; Y]$  corresponds to the homomorphism

$$\delta: C^n(K; \pi) \rightarrow C^{n+1}(K; \pi)$$

given by  $\delta f(\sigma) = \sum_\tau \deg(p_\tau f_{\partial\sigma}) f(\tau)$ , where  $\sigma$  is an  $(n+1)$ -cell and  $\tau$  ranges over the  $n$ -cells. But the right-hand side is  $f(\sum_\tau \deg(p_\tau f_{\partial\sigma}) \tau) = f(\partial\sigma)$ . Therefore,  $\delta$  is precisely the cellular coboundary up to sign, justifying our use of that symbol.

We have constructed the isomorphism

$$[K; \Omega Y] \approx H^n(K; \pi),$$

which is natural in  $K$ .

We can replace  $\Omega Y$  by a CW-complex  $L$  since the construction of a CW-complex  $L$  of type  $(\pi, n)$  in Corollary 11.9 makes it clear how to also define a weak homotopy equivalence  $L \rightarrow \Omega Y$  (or into any  $K(\pi, n)$ ). This is actually a homotopy equivalence because Milnor [1] has shown that  $\Omega Y$  has the homotopy type of a CW-complex when  $Y$  has, but we neither need nor will prove this fact. By Corollary 11.13,  $[K; L] \approx [K; \Omega Y]$  for all CW-complexes  $K$ . Replacing  $[SK; Y]$  by  $[K; \Omega Y]$  and then by  $[K; L]$ , the important part of diagram (\*) becomes

$$\begin{array}{ccccc} & & [K^{(n)}/K^{(n-1)}; L] \approx C^n(K; \pi) = \text{Hom}(C_n(K), \pi) & & \\ & & \downarrow & & \\ 0 & \longrightarrow & [K; L] & \longrightarrow & [K^{(n)}; L] \\ & & & & \downarrow \\ & & & & 0. \end{array}$$

Starting with a map  $\phi: K \rightarrow L$  representing  $[\phi] \in [K; L]$ , chasing it to  $C^n(K; \pi)$  is given by first restricting it to  $K^{(n)}$  then (or prior on  $K$ ) passing to a homotopic map that takes  $K^{(n-1)}$  to the base point of  $L$ , and then passing to the induced map  $\phi': K^{(n)}/K^{(n-1)} \rightarrow L$ . Finally, this gives a cochain  $c_{\phi'}$  on  $K$  by  $c_{\phi'}(\tau) = [\phi' \circ \bar{f}_\tau] \in \pi_n(L) = \pi$ , where  $\bar{f}_\tau: S^n \rightarrow K^{(n)}/K^{(n-1)} = \bigvee_\tau S^n$  is the inclusion of the  $\tau$ th sphere induced by the characteristic map  $f_\tau: D^n \rightarrow K^{(n)}$ . As shown,  $c_{\phi'}$  is a cocycle when it comes from  $\phi: K \rightarrow L$  this way. (One can also see that directly.) The fact that  $[\phi] \mapsto [c_{\phi'}]$  is a bijection means that the class  $[c_{\phi'}]$  depends only on  $[\phi]$  and this means that the cocycle  $c_{\phi'}$  depends on the choice of  $\phi'$ , given  $\phi$ , only up to a coboundary. (One can also see this directly, but we do not need that.)

Describing the correspondence the other direction is as easy: Starting with a class  $\xi \in H^n(K; \pi)$ , represent it by a cocycle  $c: C_n(K) \rightarrow \pi$  and construct a map

$$f_c: K^{(n)}/K^{(n-1)} = \bigvee_\tau S^n \rightarrow L$$

by putting a representative  $S^n \rightarrow L$  of  $c(\tau) \in \pi = \pi_n(L)$  on the  $\tau$ th sphere. This, then, induces a map  $K^{(n)} \rightarrow L$  and it extends to  $f: K \rightarrow L$  because  $c$  is a cocycle and by the main discussion.

If we take the space  $L$  of type  $(\pi, n)$  to be as constructed in Corollary 11.9 then  $L^{(n-1)} = \{*\}$  and so  $L^{(n)} = \bigvee_\tau S^n$  where the  $n$ -cells  $\tau$  correspond to given generators of  $\pi$ . Then it is clear that  $1 \in [L; L]$  corresponds to the class  $u \in H^n(L; \pi)$  represented by the cocycle  $c$  taking each  $n$ -cell  $\tau$  to the corresponding generator of  $\pi$ . Then  $c^*: H_n(L) \rightarrow \pi$  is an isomorphism. (Also recall that the Hurewicz map  $\pi_n(L) \rightarrow H_n(L)$  is an isomorphism.) A class  $v \in H^n(L; \pi)$  which corresponds to an isomorphism  $H_n(L) \rightarrow \pi$  is called a “characteristic class.” This is defined for *any* space with  $\pi$  as the first nonzero homotopy group.

Let us denote by  $T: [K; L] \xrightarrow{\sim} H^n(K; \pi)$  our natural equivalence of functors. Then  $T(1) = u$ . For a map  $f: K \rightarrow L$ , the commutative diagram

$$\begin{array}{ccc} [L; L] & \xrightarrow{T} & H^n(L; \pi) \\ \downarrow f^\# & & \downarrow f^* \\ [K; L] & \xrightarrow{T} & H^n(K; \pi) \end{array}$$

shows that  $f^\#(1) = [f]$  and  $T[f] = f^*(T(1)) = f^*(u)$ . More generally, any map  $f: K \rightarrow K'$  of CW-complexes induces

$$\begin{array}{ccc} [K'; L] & \xrightarrow{T} & H^n(K'; \pi) \\ \downarrow f^\# & & \downarrow f^* \\ [K; L] & \xrightarrow{T} & H^n(K; \pi). \end{array}$$

If  $f: L \rightarrow L$  is a homotopy equivalence, then  $f^*$  is an isomorphism. It follows that  $T[f] = f^*(u)$  is characteristic. Conversely, if  $f$  is such that  $f^*(u)$  is

characteristic, then  $f^\# : [L; L] \rightarrow [L; L]$  is a bijection, and so there is a map  $g : L \rightarrow L$  such that  $f^\# [g] = 1$ . This implies that  $g \circ f \simeq 1$  and hence  $f^* g^* = 1$ , so that  $g^*$  is also an isomorphism and  $f \circ g \simeq 1$ . This essentially means that any characteristic class  $u \in H^n(L; \pi)$  is as good as any other.

For any space  $Y$  of type  $(\pi, n)$  there is a weak homotopy equivalence  $L \rightarrow Y$  and this induces  $[K; L] \xrightarrow{\approx} [K; Y]$ . This allows the results for  $[K; L]$  to be transferred to  $[K; Y]$ . Summarizing, we get:

**12.1. Theorem.** *Let  $Y$  be a space of type  $(\pi, n)$ ,  $\pi$  abelian, and let  $u \in H^n(Y; \pi)$  be characteristic. Then there is a natural equivalence of functors*

$$T_u : [K; Y] \rightarrow H^n(K; \pi)$$

of CW-complexes  $K$ , given by  $T_u[f] = f^*(u)$ . □

Note that if  $(K, A)$  is a relative CW-complex then  $K/A$  is a CW-complex and so it follows that, in the situation of Theorem 12.1,

$$[K/A; Y] \approx H^n(K/A; \pi) \approx H^n(K, A; \pi).$$

There are three cases of well-known spaces of type  $(\pi, n)$ . The most obvious one is  $S^1$  which is a  $K(\mathbb{Z}, 1)$ . Also  $\mathbb{C}P^\infty = \bigcup \mathbb{C}P^n$ , with the weak topology, is a  $K(\mathbb{Z}, 2)$ . This follows from the fibrations  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$  and the fact that  $\pi_i(\mathbb{C}P^\infty) = \varinjlim \pi_i(\mathbb{C}P^n)$ . Similarly,  $\mathbb{P}^\infty$  is a  $K(\mathbb{Z}_2, 1)$ , and, more generally, an infinite lens space is a  $K(\mathbb{Z}_p, 1)$ .

Let us now discuss an application to “cohomology operations.”

**12.2. Definition.** A cohomology operation  $\theta$  of type  $(n, \pi; k, \omega)$  is a natural transformation

$$\theta : H^n(\cdot; \pi) \rightarrow H^k(\cdot; \omega)$$

of functors of CW-complexes. It need not consist of homomorphisms.

For example,  $\alpha \mapsto \alpha^2$ , for  $\alpha \in H^n(\cdot; \mathbb{Z})$  is a cohomology operation of type  $(n, \mathbb{Z}; 2n, \mathbb{Z})$ , and similarly with the higher powers and other coefficient groups. Another example is the Bockstein  $\beta_0 : H^n(\cdot; \mathbb{Z}_p) \rightarrow H^{n+1}(\cdot; \mathbb{Z})$ , which is of type  $(n, \mathbb{Z}_p; n+1, \mathbb{Z})$ . Similarly, the Bockstein  $\beta : H^n(\cdot; \mathbb{Z}_p) \rightarrow H^{n+1}(\cdot; \mathbb{Z}_p)$  is of type  $(n, \mathbb{Z}_p; n+1, \mathbb{Z}_p)$ .

**12.3. Theorem (Serre).** *There is a one-one correspondence between the cohomology operations of type  $(n, \pi; k, \omega)$  and the elements of  $H^k(K(\pi, n); \omega)$ , which is given by  $\theta \mapsto \theta(u)$  where  $u \in H^n(K(\pi, n); \pi)$  is characteristic.*

**PROOF.** This is equivalent, via Theorem 12.1, to the statement that operations

$$\psi : [X; K(\pi, n)] \rightarrow [X; K(\omega, k)]$$

correspond to elements of  $[K(\pi, n); K(\omega, k)]$  via  $\psi \mapsto \psi(1)$ . To simplify notation, let  $K = K(\pi, n)$  and  $L = K(\omega, k)$ .

Given  $f: X \rightarrow K$  we have the diagram

$$\begin{array}{ccc} [K; K] & \xrightarrow{\psi} & [K; L] \\ \downarrow f^\# & & \downarrow f^\# \\ [X; K] & \xrightarrow{\psi} & [X; L], \end{array}$$

which, on elements, is

$$\begin{array}{ccc} [1] & \mapsto & \psi(1) \\ \downarrow & & \downarrow \\ [f] & \mapsto & \psi[f]. \end{array}$$

Thus,  $\psi[f] = f^\# \psi(1) = [g \circ f]$  where  $g: K \rightarrow L$  represents  $\psi(1) \in [K; L]$ . Conversely,  $[g] \in [K; L]$  induces the operation  $\psi$ , by defining  $\psi[f] = [g \circ f]$ .  $\square$

For example, the fact that  $H^{2n}(\mathbb{CP}^\infty; \mathbb{Z}) \approx \mathbb{Z}$  implies that all cohomology operations  $\theta: H^2(\cdot; \mathbb{Z}) \rightarrow H^{2n}(\cdot; \mathbb{Z})$  have the form  $\theta(\alpha) = k\alpha^n$  for some  $k \in \mathbb{Z}$ .

On the other hand, the fact that  $\alpha \mapsto \alpha^2$  of  $H^4(X; \mathbb{Z}) \rightarrow H^8(X; \mathbb{Z})$  is nontrivial on some space  $X$  (e.g.,  $\mathbb{CP}^\infty$ ) implies that  $H^8(K(\mathbb{Z}, 4); \mathbb{Z}) \neq 0$ .

Similarly, the fact that  $H^2(\mathbb{P}^\infty; \mathbb{Z}) \approx \mathbb{Z}_2$  implies that there is exactly one nontrivial operation  $H^1(\cdot; \mathbb{Z}_2) \rightarrow H^2(\cdot; \mathbb{Z})$ . Since the Bockstein in that case is nontrivial (e.g., on  $\mathbb{P}^2$ ), it is that unique operation.

#### 12.4. Corollary. No nontrivial cohomology operation lowers dimension.

**PROOF.** This follows from the fact that  $H^k(K(\pi, n); \omega) = 0$  for  $0 < k < n$  by the Hurewicz and Universal Coefficient Theorems, or simply by the construction of  $K(\pi, n)$  in Corollary 11.9, which has trivial  $(n-1)$ -skeleton.  $\square$

In the next section we will need some technical items about connections between characteristic elements, and another matter. This will fill out the remainder of this section. It is suggested that a first time reader skip this material and refer back to the statements, which are quite believable, when they are used in the following section.

In the remainder of this section, and in the following sections, we shall make the blanket assumption that all pointed spaces under consideration are *well-pointed*.

Let the “suspension isomorphism” in cohomology be defined as the composition

$$S: \tilde{H}^n(X) \xrightarrow{\delta^*} H^{n+1}(CX, X) \xleftarrow{\approx} \tilde{H}^{n+1}(SX) \approx H^{n+1}(SX)$$

(for  $n \geq 0$ ). Sometimes this is defined with a difference in sign. This would have no effect on our main formulas, just on some details of the derivations.

We also use the analogous definition for the suspension isomorphism in homology and the suspension homomorphism for homotopy groups.

**12.5. Lemma.** *If  $f: X \rightarrow Y$  is a map between  $(n-1)$ -connected spaces which induces an isomorphism on  $\pi_n(X) \rightarrow \pi_n(Y) \approx \pi$  and if  $u \in H^n(Y; \pi)$  is characteristic, then  $f^*(u) \in H^n(X; \pi)$  is characteristic.*

**PROOF.** There is the commutative diagram

$$\begin{array}{ccc} H^n(Y; \pi) & \xrightarrow[\approx]{\beta_Y} & \text{Hom}(H_n(Y), \pi) \\ \approx \downarrow f^* & & \approx \downarrow \text{Hom}(f_*, 1) \\ H^n(X; \pi) & \xrightarrow[\approx]{\beta_X} & \text{Hom}(H_n(X), \pi) \end{array}$$

where the  $\beta$ 's are the maps in the Universal Coefficient Theorem (Theorem 7.2 of Chapter V). By definition,  $u \in H^n(Y; \pi)$  is characteristic  $\Leftrightarrow \beta_Y(u): H_n(Y) \rightarrow \pi$  is an isomorphism. We have that  $\beta_X(f^*(u))(a) = \beta_Y(u)(f_*(a))$  by commutativity. Thus  $\beta_X(f^*(u)) = \beta_Y(u) \circ f_*$  is an isomorphism, implying that  $f^*(u)$  is characteristic.  $\square$

**12.6. Lemma.** *The class  $u \in H^n(Y; \pi)$  is characteristic, where  $Y$  is  $(n-1)$ -connected,  $\Leftrightarrow Su \in H^{n+1}(SY; \pi)$  is characteristic.*

**PROOF.** The Hurewicz Theorem implies that  $SY$  is  $n$ -connected. It is an immediate consequence of the definition that the following diagram commutes up to sign (which can be seen to be  $(-1)^{n+1}$ ):

$$\begin{array}{ccc} H^n(Y; \pi) & \xrightarrow[\approx]{\beta_Y} & \text{Hom}(H_n(Y), \pi) \\ \approx \downarrow S & & \approx \uparrow \text{Hom}(S, 1) \\ H^{n+1}(SY; \pi) & \xrightarrow[\approx]{\beta_{SY}} & \text{Hom}(H_{n+1}(SY), \pi). \end{array}$$

Then  $\beta_{SY}(S(u))(Sa) = \pm \beta_Y(u)(a)$  and so  $\beta_{SY}(S(u)) = \pm \beta_Y(u) \circ S^{-1}$  is an isomorphism.  $\square$

**12.7. Lemma.** *The diagram*

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{S} & \pi_{n+1}(SX) \\ \downarrow & & \downarrow \\ H_n(X) & \xrightarrow{S} & H_{n+1}(SX) \end{array}$$

*commutes, where the verticals are the Hurewicz maps.*

**PROOF.** The suspension for homotopy is defined as the composition along

the top of the commutative diagram

$$\begin{array}{ccccc}
 \pi_n(X) & \xleftarrow{\approx} & \pi_{n+1}(CX, X) & \longrightarrow & \pi_{n+1}(SX) \\
 \downarrow & & \downarrow & & \downarrow \\
 H_n(X) & \xleftarrow{\approx} & H_{n+1}(CX, X) & \xrightarrow{\approx} & H_{n+1}(SX, *)
 \end{array}$$

and the lemma follows.  $\square$

For any space  $K$  consider the map  $\lambda: S\Omega K \rightarrow K$  which is the adjoint to  $1: \Omega K \rightarrow \Omega K$ . That is,  $\lambda$  is induced by the evaluation map  $K^I \times I \rightarrow K$ . The class  $[\lambda]$  corresponds to  $[1]$  under the bijection  $[S\Omega K; K] \leftrightarrow [\Omega K; \Omega K]$ . The diagram (of sets)

$$\begin{array}{ccc}
 (K^I)^X & & \\
 \downarrow & \searrow & \\
 (K^I \times I)^{X \times I} & \longrightarrow & K^{X \times I}
 \end{array}$$

commutes where the horizontal map is induced by the evaluation, the diagonal one is the exponential correspondence  $f'(x, t) = f(x)(t)$ , and the vertical map is  $f \mapsto f \times 1$  where  $(f \times 1)(x, t) = (f(x), t)$ . This induces the diagram

$$\begin{array}{ccc}
 [X; \Omega K] & & \\
 S \downarrow & \searrow \approx & \\
 [SX; S\Omega K] & \xrightarrow{\lambda_{\#}} & [SX; K],
 \end{array}$$

where the diagonal is the adjoint (exponential) correspondence. Thus this diagram commutes.

Now if  $K = K(\pi, n+1)$  then we conclude that the diagram

$$\begin{array}{ccccc}
 H_n(\Omega K) & \approx & [S^n; \Omega K] & & \\
 S \downarrow \approx & & \downarrow & \searrow \approx & \\
 H_{n+1}(S\Omega K) & \approx & [S^{n+1}; S\Omega K] & \xrightarrow{\lambda_{\#}} & [S^{n+1}; K]
 \end{array}$$

commutes and it follows that

$$\lambda_{\#}: \pi_{n+1}(S\Omega K) \xrightarrow{\approx} \pi_{n+1}(K)$$

is an isomorphism.

Now choose any characteristic class  $u \in H^{n+1}(K; \pi)$ . By Lemma 12.5,  $\lambda^*u \in H^{n+1}(S\Omega K; \pi)$  is characteristic. By Lemma 12.6,  $v = S^{-1}\lambda^*u \in H^n(\Omega K; \pi)$  is characteristic. These remarks imply the following result:

**12.8. Proposition.** *Let  $K = K(\pi, n+1)$  and let  $u \in H^{n+1}(K; \pi)$  be characteristic. Then  $\lambda^*u \in H^{n+1}(S\Omega K; \pi)$  and  $v = S^{-1}\lambda^*u \in H^n(\Omega K; \pi)$  are characteristic and*

the following diagram commutes:

$$\begin{array}{ccccc}
 [A; \Omega K] & \xrightarrow{S} & [SA; S\Omega K] & \xrightarrow{\lambda_{\#}} & [SA; K] \\
 \approx \downarrow T_v & & \approx \downarrow T_{\lambda^* u} & \approx \swarrow T_u & \\
 H^n(A; \pi) & \xrightarrow{S} & H^{n+1}(SA; \pi) & & 
 \end{array}
 \quad \square$$

**12.9. Proposition.** For a cofibration  $A \hookrightarrow X$  let  $c: X \cup CA \rightarrow SA$  be the collapsing map. Then the composition (for arbitrary coefficients)

$$H^n(A) \xrightarrow{S} \tilde{H}^{n+1}(SA) \xrightarrow{c^*} \tilde{H}^{n+1}(X \cup CA) \xrightarrow{\sim} H^{n+1}(X, A)$$

is  $-\delta^*$ , where  $\delta^*$  is the connecting homomorphism for the exact sequence of  $(X, A)$ .

**PROOF.** Consider the diagram

$$\begin{array}{ccccc}
 H^n(A) & \xrightarrow{\delta^*} & H^{n+1}(CA, A) & \xleftarrow{\sim} & H^{n+1}(SA, *) \\
 \uparrow \approx & & \uparrow \approx & & \parallel \\
 H^n(A \cup *, *) & \xrightarrow{\delta^*} & H^{n+1}(CA, A \cup *) & \xleftarrow{\sim} & H^{n+1}(SA, *) \\
 \downarrow \approx & & \downarrow \approx & & \parallel \\
 H^n(A \times \partial I, A \times \{1\}) & \xrightarrow{\delta^*} & H^{n+1}(A \times I, A \times \partial I) & \xleftarrow{\sim} & H^{n+1}(SA, *) \\
 \downarrow \approx & & \downarrow -1 & & \downarrow -1 \\
 H^n(A \times \partial I, A \times \{0\}) & \xrightarrow{\delta^*} & H^{n+1}(A \times I, A \times \partial I) & \xleftarrow{\sim} & H^{n+1}(SA, *) \\
 \uparrow \approx & & \uparrow \approx & & \downarrow \approx \\
 H^n(A \times \{1\} \cup X \times \{0\}, X \times \{0\}) & \xrightarrow{\delta^*} & H^{n+1}(A \times I \cup X \times \{0\}, A \times \{1\} \cup X \times \{0\}) & \xleftarrow{\sim} & H^{n+1}(S \cup CA, X) \\
 \downarrow \approx & & \downarrow & & \downarrow \\
 H^n(A \times \{1\}) & \xrightarrow{\delta^*} & H^{n+1}(A \times I \cup X \times \{0\}, A \times \{1\}) & \xleftarrow{\sim} & H^{n+1}(X \cup CA, *) \\
 \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\
 H^n(A \times I) & \xrightarrow{\delta^*} & H^{n+1}(A \times I \cup X \times \{0\}, A \times I) & \xleftarrow{\sim} & H^{n+1}(X \cup CA, CA) \\
 \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
 H^n(A) & \xrightarrow{\delta^*} & H^{n+1}(X, A) & \xleftarrow{\sim} & H^{n+1}(X, A)
 \end{array}$$

Some of the  $\delta^*$  maps in the diagram are from exact sequences of triples. The horizontal isomorphisms on the right are induced by obvious maps as are the vertical homomorphisms. The composition along the left is the identity and so, from the upper left, all the way down and then right to  $H^{n+1}(X, A)$  is just  $\delta^*$ . The composition along the top is  $S$ , by definition. The composition from the upper right, all the way down and then left to  $H^{n+1}(X, A)$  is  $-c^*$ , the sign caused by the inversion of the parameter  $SA \rightarrow SA$  midway down. Hence  $c^* \circ S = -\delta^*$  as claimed.  $\square$

**12.10. Lemma.** Let  $i_0, i_1: X \rightarrow X \times \partial I$  be  $i_0(x) = (x, 0)$  and  $i_1(x) = (x, 1)$ . Then for  $\delta^*: H^n(X \times \partial I) \rightarrow H^{n+1}(X \times I, X \times \partial I) \approx H^{n+1}(SX)$ , with any coefficients, we have  $S^{-1}\delta^* = i_0^* - i_1^*$ .

**PROOF.** We know that  $(i_0^*, i_1^*): H^n(X \times \partial I) \xrightarrow{\sim} H^n(X) \oplus H^n(X)$ . Let  $j_0, j_1: H^n(X) \rightarrow H^n(X \times \partial I)$  induce the inverse isomorphism, so that  $i_0^* j_0 = 1 = i_1^* j_1$

and  $i_0^* j_1 = 0 = i_1^* j_0$ . Then  $j_0 i_0^* + j_1 i_1^* = 1$ . Clearly,  $j_0$  is the composition

$$j_0: H^n(X) \xleftarrow{\approx} H^n(X \times \partial I, X \times \{1\}) \xrightarrow{h^*} H^n(X \times \partial I)$$

induced by  $x \mapsto (x, 0)$  and the inclusion  $h: (X \times \partial I, \emptyset) \hookrightarrow (X \times \partial I, X \times \{1\})$ . Also  $j_1 = \omega^* j_0$  where  $\omega$  is the reversal of the  $I$  parameter. Consider the following commutative diagram, similar to that in the proof of Proposition 12.9:

$$\begin{array}{ccccc}
 H^n(X) & \xrightarrow{\delta^*} & H^{n+1}(CX, X) & \xleftarrow{\approx} & H^{n+1}(SX) \\
 \uparrow \approx & & \uparrow \approx & & \parallel \\
 H^n(X \cup *, *) & \xrightarrow{\delta^*} & H^{n+1}(CX, X \cup *) & \xleftarrow{\approx} & H^{n+1}(SX) \\
 \downarrow \approx & & \downarrow \approx & & \parallel \\
 H^n(X \times \partial I, X \times \{1\}) & \xrightarrow{\delta^*} & H^{n+1}(X \times I, X \times \partial I) & \xleftarrow{\approx} & H^{n+1}(SX) \\
 \downarrow h^* & & \parallel & & \\
 H^n(X \times \partial I) & \xrightarrow{\delta^*} & H^{n+1}(X \times I, X \times \partial I) & & 
 \end{array}$$

This shows that  $S^{-1} \delta^* j_0 = 1$ , since  $S$  is the composition from top left to bottom right, going right then down. Then  $S^{-1} \delta^* j_1 = S^{-1} \delta^* \omega^* j_0 = S^{-1} \omega^* \delta^* j_0 = -S^{-1} \delta^* j_0 = -1$ , since  $\omega$  induces  $-1$  on  $H^*(SX)$ . Consequently,  $S^{-1} \delta^* = S^{-1} \delta^* \circ 1 = S^{-1} \delta^* (j_0 i_0^* + j_1 i_1^*) = i_0^* - i_1^*$ .  $\square$

### PROBLEMS

1. Show that any  $K(\mathbb{Z}, n)$  is infinite dimensional for each even  $n > 0$ .
2. Show that any  $K(\mathbb{Z}_2, n)$  is infinite dimensional for each  $n > 0$ .
3. Show that there are no nontrivial cohomology operations of type  $(1, \mathbb{Z}; k, \omega)$  for any  $k > 1$  and any  $\omega$ .
4.  $\blacklozenge$  Show that there are no nontrivial cohomology operations of type  $(n, \mathbb{Z}; n+1, \omega)$  for any  $n > 0$  and any  $\omega$ .
5. Rederive Hopf's Classification Theorem (Theorem 11.6 of Chapter V) as a corollary of the results of this section. (*Hint:* Use Corollary 11.13 and Theorem 11.8.)

## 13. Obstruction Theory $\odot$

In this section and the next we impose the blanket assumption that all pointed spaces under consideration are *well-pointed*. This is not an important restriction and is made merely to avoid having to distinguish between reduced and unreduced suspensions.