Review questions for the Final Exam, Spring 2013

1. Basic spaces: $\mathbf{R}^n$, S^n, stereographic projection. The space S^∞.

2. Projective spaces $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$: definitions, local coordinate system, the Hopf maps $S^n \to \mathbf{RP}^n$, $S^{2n+1} \to \mathbf{CP}^n$, $S^{4n+3} \to \mathbf{HP}^n$.

3. Prove the homeomorphisms: $\mathbf{RP}^1 \cong S^1$, $\mathbf{CP}^1 \cong S^2$, $\mathbf{HP}^1 \cong S^4$.

4. Prove that $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$ are connected and compact spaces.

5. Define Grassmannian manifolds $G(n,k)$, $\mathbf{C}G(n,k)$, in particular, construct local coordinate systems and find their dimensions.

6. Prove that the Grassmannian manifolds $G(n,k)$ and $\mathbf{C}G(n,k)$ are compact and connected.

7. Define classic Lie groups $GL(\mathbf{R}^k)$, $GL(\mathbf{C}^k)$, $O(k)$, $SO(k)$, $U(k)$, $SU(k)$. Prove that the spaces $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact. How many connected components does each of these spaces have?

8. Prove that $SO(2)$ and $U(1)$ are homeomorphic to S^1, $SO(3)$ is homeomorphic to $\mathbf{RP}^3$, and $SU(2)$ is homeomorphic to S^3.

9. Prove that $SO(4) \cong SO(3) \times S^3$.

10. Define Stiefel manifolds $V(n,k)$, $\mathbf{C}V(n,k)$, $\mathbf{H}V(n,1)$. Prove the following homeomorphisms:

$$\begin{gathered} V(n,n) \cong O(n), \quad V(n,n-1) \cong SO(n), \\ \mathbf{C}V(n,n) \cong U(n), \quad \mathbf{C}V(n,n-1) \cong SU(n), \\ V(n,1) \cong S^{n-1}, \quad \mathbf{C}V(n,1) \cong S^{2n-1}, \quad \mathbf{H}V(n,1) \cong S^{4n-1}. \end{gathered}$$

11. Define action of the groups $O(k)$, $U(k)$ on the Stiefel manifolds $V(n,k)$, $\mathbf{C}V(n,k)$. Prove the following homeomorphisms: $V(n,k)/O(k) \cong G(n,k)$, $\mathbf{C}V(n,k)/U(k) \cong \mathbf{C}G(n,k)$.

12. Prove the following homeomorphisms:

$$\begin{gathered} S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1), \\ S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1), \\ G(n,k) \cong O(n)/O(k) \times O(n-k), \quad \mathbf{C}G(n,k) \cong U(n)/U(k) \times U(n-k). \end{gathered}$$

13. Prove that the Klein bottle Kl^2 is homeomorphic to the union of two Mëbius bands along the circle.

14. Prove that $Kl^2 \# \mathbf{RP}^2$ is homeomorphic to $\mathbf{RP}^2 \# T^2$.

15. Define a cylinder and a cone of a map $f : X \to Y$. Prove that the cones of the maps $c : S^n \to \mathbf{RP}^n$ and $h : S^{2n+1} \to \mathbf{CP}^n$ are homeomorphic to $\mathbf{RP}^{n+1}$ and $\mathbf{CP}^{n+1}$ respectively.

16. Define suspension. Prove that $\Sigma(S^n) \cong S^{n+1}$.

17. Define a compact-open topology on $\mathcal{C}(X,Y)$. Prove the homeomorphism: $\mathcal{C}(X, \mathcal{C}(Y,Z)) \cong \mathcal{C}(X \times Y, Z)$ provided appropriate restrictions on X and Y. Prove that this homeomorphism is natural.

18. Define the spaces of paths $\mathcal{E}(X, x_0, x_1)$, $\mathcal{E}(X, x_0)$, and loops $\Omega(X, x_0)$. Prove that the spaces $\Omega(S^n, x_0)$ and $\Omega(S^n, x_1)$ are homeomorphic for any points $x_0, x_1 \in S^n$.

19. Let X, Y be pointed spaces. Prove the homeomorphism $\mathcal{C}(\Sigma(X), Y) \cong \mathcal{C}(X, \Omega(Y))$. for Hausdorff and locally compact spaces X, Y, Z. Prove that this homeomorphism is natural.

20. Define smash-product $X \wedge Y$. Prove that $S^n \wedge S^k \cong S^{n+k}$ (as pointed spaces).

21. Define homotopy of two maps. Prove that the maps $\phi^* : [X', Y] \to [X, Y]$, $\psi_* : [X, Y] \to [X, Y']$ induced by maps $\phi : X \to X'$, $\psi : Y \to Y'$ are well-defined.

22. Give three definitions of homotopy equivalence. Prove that they are equivalent.

23. Prove that $X \sim Y$ implies $\Sigma(X) \sim \Sigma(Y)$ and $\Omega(X) \sim \Omega(Y)$.

24. Give a definition of a contractible space. Prove that $\mathcal{E}(X, x_0)$ is a contractible.

25. Prove that a space X is contractible if and only if it is homotopy equivalent to a point.

26. Prove that a space X is contractible if and only if every map $f : Y \to X$ is null-homotopic.

27. Give definition of a retract and deformational retract. Examples. Prove that $\{0\} \cup \{1\}$ is not a retact of $I = [0, 1]$. Define map of pairs. Examples.

28. Define a CW-complex. Give examples of cell decomposition. Show that the axiom (W) does not imply the axiom (C) and wise-versa.

29. Construct a cellular decomposion of the wedge $X = S^1 \vee S^2$ (with a single 2-cell e^2) such that a closure of the cell e^2 is not a CW-subcomplex of X.

30. Construct a cellular decomposion of the wedge $X = \Sigma(S^n \vee S^k)$. Prove that $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$.

31. Prove that a CW-complex compact if and only if it is finite.

32. Construct a cellular decomposition of S^n, D^n, $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$.

33. Construct a cellular decomposition of the oriented 2-manifold of genus g.

34. Define the Schubert cells $e(\sigma)$ corresponding to the Schubert symbol σ. Give examples.

35. Define the spaces H^j, $\overline{H}^j$. Prove that a k-plane π belongs to $e(\sigma)$ if and only if there exists its basis $v_1, \ldots, v_k$, such that $v_1 \in H^{\sigma_1}, \ldots, v_k \in H^{\sigma_k}$.

36. Prove the following statement: Let $\pi \in e(\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$. Then there exists a unique orthonormal basis $v_1, \ldots, v_k$ of π, so that $v_1 \in H^{\sigma_1}, \ldots, v_k \in H^{\sigma_k}$.

37. Define the sets $E(\sigma), \overline{E}(\sigma) \subset V(n, k)$. Prove that the set $\overline{E}(\sigma) \subset V(n, k)$ is homeomorphic to the closed cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_k - k)$. Furthermore the map $q : e(\sigma) \to E(\sigma)$ is a homeomorphism.

38. Define the transformations $T_{u,v}$, prove its properties. Explain how the transformations $T_{u,v}$ are used to prove that $\overline{E}(\sigma) \subset V(n, k)$ is homeomorphic to a closed cell of dimension $d(\sigma)$.

39. Prove the statement: a collection of $\begin{pmatrix} k \\ n \end{pmatrix}$ Schubert cells $e(\sigma)$ gives $G(n, k)$ a cell-decomposition.

40. Define when a pair (X, Y) is a Borsuk pair. Prove that a CW-pair (X, Y) is a Borsuk pair (in the case when X, Y are finite complexes).

41. Let (X, A) be a Borsuk pair. Prove that A is a deformation retract of X if and only if the inclusion $A \to X$ is a homotopy equivalence.

42. Prove the statement: let X be a CW-complex and $A \subset X$ be its contractible subcomplex. Then X is homotopy equivalent to the complex X/A.

43. Prove that for a CW-pair (X, A) $X/A \sim X \cup C(A)$.

44. State Cellular Approximation Theorem. Prove it using Free Point Lemma.

45. State and prove Free Point Lemma.

46. Define homotopy groups $\pi_n(X)$. Prove that $\pi_n(X)$ is commutative group for $n \geq 2$. Prove that $\pi_k(S^n)$ is a trivial group for $k < n$.

47. Prove the satement: Let X be a CW-complex with only one zero-cell and without cells of dimension $q < n$, and Y be a CW-complex of dimension $< q$. Then any map $Y \to X$ is homotopic to a constant map.

48. Define n-connected space. Prove the statement: Any n-connected CW-complex homotopy equivalent to a CW-complex with a single zero cell and without cells of dimensions $1, 2, \ldots, n$.

49. Prove that if $f, g : X \to Y$ are homotopic maps, than the homomorphisms $f_*, g_* : \pi_n(X) \to \pi_n(Y)$ coincide.

50. Prove that if X is a path-connected space, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. Describe all isomorphisms here.

51. Prove that $\pi_1 S^1 \cong \mathbf{Z}$.

52. Prove that $\pi_1(\bigvee_{\alpha \in A} S^1_\alpha)$ is a free group.

53. Prove that $\pi_1(X, x_0) \cong \pi_1(X^{(2)}, x_0)$, where X is a connected CW-complex and $X^{(2)}$ its 2-skeleton.

54. Compute $\pi_1(M^2)$ for two-dimensional oriented closed manifold of genus g, the sphere with g handles.

55. Compute $\pi_1(M^2)$ for two-dimensional non-oriented closed manifold of genus g, the projective plane or the Klein bottle with g handles.

56. Let $M = \mathbf{RP}^2 \# \cdots \# \mathbf{RP}^2$ (n times). Compute $\pi_1(M)$.

57. Compute $\pi_1(\mathbf{RP}^2 \# \mathbf{RP}^2)$ and $\pi_1(Kl^2 \# \mathbf{RP}^2)$.

58. Define $G_1 * G_2$. Give examples. Prove that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

59. Define $G_1 *_H G_2$. Give examples. State and prove Van Kampen Theorem.

60. Define covering space. Give examples. Construct n-fold covering of $S^1 \vee S^1$ (including $n = \infty$).

61. State and prove Theorem on Covering Homotopy.

62. Prove that covering $p : T \to X$ induces a monomorphism $p_* : \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$.

63. Prove that a loop $\alpha_1 \cdots \alpha_k$, where α_j is a loop going along the j-th circle in the wedge $\bigvee_{j=1}^k S^1_j$, is not homotopic to zero.

64. Let $p : T \to X$ be a covering, and $f, g : Z \to T$ be two maps so that $p \circ f = p \circ g$, where Z is path-connected. Assume that $f(z) = g(z)$ for some point $z \in Z$. Prove that $f = g$.

65. Prove that $\pi_k(\mathbf{RP}^n) = 0$ if $1 < k < n$.

66. Prove that any map $f : \mathbf{RP}^2 \to S^1$ is homotopic to a constant map.

67. Let Kl^2 be the Klein bottle. Construct two-folded covering space $Kl^2 \to T^2$. Compute $\pi_n(Kl^2)$ for all n.

68. Let $p : T \to X$ be a covering, $p(\tilde{x}_0) = x_0$. Prove that there is one-to-one correspondence

$$\pi_1(X(X, x_0)/p_*(\pi_1(T, \tilde{x}_0)) \Longleftrightarrow p^{-1}(x_0).$$

Prove that $p^{-1}(x_0) \cong p^{-1}(x_1)$ for any points $x_0, x_1 \in X$.

69. Let p : $T \to X$ be a covering map, and let $\Gamma = p^{-1}(x_0)$. Prove that Γ is a transitive right G-set for $G = \pi_1(X, x_0)$.

70. Let X by "good" space and $G = \pi_1(X, x_0)$. Prove that athere is a bijection between isomorphism classes of covering spaces of X and transitive right G-sets given by

$$\{p : Y \to X\} \mapsto p^{-1}(x_0).$$

71. Let $p : T \to X$ be a covering, and $f : Z \to X$ be a map, $f(z_0) = x_0$, and $\tilde{x}_0 \in T$ so that $p(\tilde{x}_0) = x_0$ (here Z is path-connected space). Prove that there exists a lifting $\widetilde{f} : Z \to T$ of the map f so that $\widetilde{f}(z_0) = \tilde{x}_0$ if and only if $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \tilde{x}_0))$.

72. Define morphism of two covering spaces $T_1 p_1 \xrightarrow{p_1} X$ and $T_2 \xrightarrow{p_2} X$. Prove that two morphisms $\phi, \phi' : T_1 \to T_2$ coincide if there is a point $\tilde{x} \in T_1$ so that $\phi(\tilde{x}) = \phi'(\tilde{x})$.

73. Define a group of automorphisms (deck transformations) $\mathrm{Aut}(T \xrightarrow{p} X)$ of a covering $p : T \to X$. Prove that the group $\mathrm{Aut}(T \xrightarrow{p} X)$ acts on T without fixed points.

74. Let $p : T \to X$ be a covering, $p(\tilde{x}_0) = p(\tilde{x}_0') = x_0$, where $\tilde{x}_0 \neq \tilde{x}_0'$. Prove that there exists an automorphism $\phi \in \mathrm{Aut}(T \xrightarrow{p} X)$ such that $\phi(\tilde{x}_0) = \tilde{x}_0'$ if and only if $p_*(\pi_1(T, \tilde{x}_0)) = p_*(\pi_1(T, \tilde{x}_0'))$.

75. Prove the following statement: Two covering spaces $T_1 \xrightarrow{p_1} X$, $T_2 \xrightarrow{p_2} X$ are isomorphic if and only if for any two points $\widetilde{x}_1, \widetilde{x}_2 \in T$ such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$ the groups $(p_1)_*(\pi_1(T_1, \widetilde{x}_1))$, $(p_2)_*(\pi_1(T_2, \widetilde{x}_2))$ belong to the same conjugacy class in $\pi_1(X, x)$.

76. Let $N(H)$ be a normalizer for a subgroup H of G. Prove the following statement: Let $p : T \to X$ be a covering space. Then the group of automorphisms of this covering space is isomorphic to the group $N(p_*(\pi_1(T, \widetilde{x}_0)))/p_*(\pi_1(T, \widetilde{x}_0))$.

77. Define universal covering space over X. Prove the following statement: Let X be a path-connected CW-complex, $x_0 \in X$. Then for any subgroup $G \subset \pi_1(X, x_0)$ there exists a covering $T \xrightarrow{p} X$ and a point $\widetilde{x}_0 \in T$ so that $p_*(\pi_1(T, \widetilde{x}_0)) = G$.

78. Define homotopy groups $\pi_n(X, x_0)$, in particular define the group operation and inverse. Prove that the groups $\pi_n(X, x_0)$ are abelian if $n \geq 2$.

79. Prove that $\pi_n(X \times Y, x_0 \times y_0) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$. Compute $\pi_n(T^k)$ for all n.

80. Let X be a path-connected space, and $x_0, x_1 \in X$ be two different points. Let $\gamma : I \to X$ be a path so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Define a homomorphism $\gamma_\# : \pi_n(X, x_0) \to \pi_n(X, x_1)$. Prove that $\gamma_\#$ is an isomorphism.

81. Let M_g^2 be a two-dimensional surface of genus $g \geq 1$ (oriented). Compute the homotopy groups $\pi_q(M_g^2)$.

82. Define relative homotopy groups $\pi_n(X, A; x_0)$. Describe the group operation and the inverse element. Prove that the group $\pi_n(X, A; x_0)$ is commutative for $n \geq 3$.

83. Define the homomorphisms in the following sequence:

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A; x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \tag{1}$$

Prove that the sequence (1) is exact.

84. Let $A \subset X$ be a retract. Prove that

- $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is monomorphism,
- $j_* : \pi_n(X, x_0) \to \pi_n(X, A; x_0)$ is epimorphism,
- $\partial : \pi_n(X, A; x_0) \to \pi_{n-1}(A, x_0)$ is zero homomorphism.

85. Let A be contractible in X. Prove that

- $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is zero homomorphism,
- $j_* : \pi_n(X, x_0) \to \pi_n(X, A; x_0)$ is monomorphism,
- $\partial : \pi_n(X, A; x_0) \to \pi_{n-1}(A, x_0)$ is epimorphism.

86. State and prove Five-Lemma.

87. Let $0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$ be an exact sequence of finitely generated abelian groups. Prove that $\sum_{i=1}^{n}(-1)^i \text{rank } A_i = 0$.

88. Define locally trivial fiber bundle. Give several examples of non-trivial fiber bundles.

89. Prove that any locally–trivial fiber bundle over the cube I^q is trivial.

90. Define the covering homotopy property. Outline a proof that the covering homotopy property holds for a locally-trivial fiber bundle $E \longrightarrow B$.

91. Define a Serre fiber bundle. Let Y be an arbitrary path-connected space, $\mathcal{E}(Y, y_0)$ be the space of paths starting at y_0. Prove that the map $p : \mathcal{E}(Y, y_0) \longrightarrow Y$, where $p(s : I \longrightarrow Y) = s(1) \in Y$ is a Serre fiber bundle.

92. Let $A \subset X$, and (X, A) be a Borsuk pair (for example, a CW-pair). Let $E = \mathcal{C}(X, Y)$, $B = \mathcal{C}(A, Y)$, and the map $p : E \longrightarrow B$ be defined as $p(f : X \longrightarrow Y) = (f|_A : A \longrightarrow Y)$. Prove that the map $p : E \longrightarrow B$ is a Serre fiber bundle.

93. Define weak homotopy equivalence. Prove that finite CW-complexes X, Y are weak homotopy equivalent if and only if they are homotopy equivalent.

94. Let $p : E \longrightarrow B$ be Serre fiber bundle, where B be a path-connected space. Prove that the fibers $F_0 = p^{-1}(x_0)$ and $F_1 = p^{-1}(x_1)$ are weak homotopy equivalent for any two points $x_0, x_1 \in B$.

95. Prove that for any continuous map $f : X \longrightarrow Y$ there exists homotopy equivalent map $f_1 : X_1 \longrightarrow Y_1$, such that $f_1 : X_1 \longrightarrow Y_1$ is Serre fiber bundle.

96. Let $f : X \longrightarrow Y$ be a continuous map. Prove that there exists a homotopy equivalent map $g : X \longrightarrow Y'$, so that g is an inclusion.

97. Let $p : E \longrightarrow B$ be Serre fiber bundle, $y \in E$ be any point, $x = p(y)$, $F = p^{-1}(x)$. Prove that the homomorphism

$$p_* : \pi_n(E, F; y) \longrightarrow \pi_n(B, x)$$

is an isomorphism for all $n \geq 1$.

98. Apply the homotopy exact sequence of Serre fibration to prove that (a) $\pi_2(S^2) = \pi_1(S^1) = \mathbf{Z}$; (b) $\pi_n(S^3) = \pi_n(S^2)$.

99. Let $S^\infty \longrightarrow \mathbf{CP}^\infty$ be the Hopf fibration. Using the fact $S^\infty \sim *$, prove that $\pi_n(\mathbf{CP}^\infty) = 0$ for $n \neq 2$, and $\pi_2(\mathbf{CP}^\infty) = \mathbf{Z}$.

100. Prove that $\pi_n(\Omega(X)) \cong \pi_{n+1}(X)$ for any X and $n \geq 0$.

101. Prove that if the groups $\pi_*(B)$, $\pi_*(F)$ are finite (finitely generated), then the groups $\pi_*(E)$ are finite (finitely generated) as well.

102. Assume that a fiber bundle $p : E \longrightarrow B$ has a *section*, i.e. a map $s : B \longrightarrow E$, such that $p \circ s = Id_B$. Prove the isomorphism $\pi_n(E) \cong \pi_n(B) \oplus \pi_n(F)$.

103. State the Freudenthal Theorem. Give a detailed proof that Σ is an isomorphism.

104. Let $K, L \subset \mathbf{R}^p$ be two finite simplicial complexes fo dimensions k, l respectively. Let $k + l + 1 < p$. Prove that the simplicial complexes K and L are not linked.

105. Prove that $\pi_n(S^n) \cong \mathbf{Z}$ for each $n \geq 1$.

106. Prove that $\pi_3(S^2) \cong \mathbf{Z}$, and the Hopf map $S^3 \longrightarrow S^2$ is a representative of the generator of $\pi_3(S^2)$.

107. Define Whitehead product. State basic properties. Prove that if $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$ then $[\alpha, \beta] = (-1)^{nk}[\beta, \alpha]$.

108. Define the element $w \in \pi_{n+k-1}(S^n \vee S^k)$. Prove that the element $w \in \pi_{n+k-1}(S^n \vee S^k)$ has infinite order.

109. Prove that the element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in a kernel of each of the following homomorphisms:

(1) $i_* : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n \times S^k)$,

(2) $pr_*^{(n)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^n)$,

(3) $pr_*^{(k)} : \pi_{n+k-1}(S^n \vee S^k) \longrightarrow \pi_{n+k-1}(S^k)$.

110. Prove that the element $w \in \pi_{n+k-1}(S^n \vee S^k)$ is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \longrightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

111. Prove the isomorphism

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

112. Let $\alpha \in \pi_n(X)$, $\beta \in \pi_k(X)$. Prove that $[\alpha, \beta] \in \text{Ker } \Sigma$, where

$$\Sigma : \pi_{n+k-1}(X) \longrightarrow \pi_{n+k}(\Sigma X)$$

is the suspension homomorphism.

113. Let $\iota_{2q} \in \pi_{2q}(S^{2q})$ be a generator represented by the identity map $S^{2q} \longrightarrow S^{2q}$. Prove that the Whitehead product $[\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q})$ is a nontrivial element of infinite order.

114. Prove that the suspension $\Sigma(S^n \times S^k)$ is homotopy equivalent to the wedge $S^{n+1} \vee S^{k+1} \vee S^{n+k+1}$.

115. Outline a proof of the following statement:
Let X be a connected space (not necessarily a CW-complex) with a base point $x_0 \in X$, $f : S^n \longrightarrow X$ be a map such that $f(s_0) = x_0$, where s_0 is a base point of S^n. Let $Y = X \cup_f D^{n+1}$, and $i : X \longrightarrow Y$ be the inclusion. Then the induced homomorphism $i_* : \pi_q(X, x_0) \longrightarrow \pi_q(Y, x_0)$

(1) is an isomorphism if $q < n$,

(2) is an epimorphism if $q = n$, and

(3) the kernel Ker $i_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, x_0)$ is generated by $\gamma^{-1}[f]\gamma \in \pi_n(X, x_0)$ where $\gamma \in \pi_1(X, x_0)$.

116. Let X be an n-connected CW-complex, and Y be a k-connected CW-complex. Prove that

- $\pi_q(X \vee Y) \cong \pi_q(X) \oplus \pi_q(Y)$ if $q \leq n + k$;
- the group $\pi_q(X \vee Y)$ contains a subgroup $\pi_q(X) \oplus \pi_q(Y)$ as a direct summand.

117. Let X be an n-connected CW-complex, and Y be a k-connected CW-complex. Prove that

$$\pi_{n+k+1}(X \vee Y) \cong \pi_{n+k+1}(X) \oplus \pi_{n+k+1}(Y) \oplus [\pi_n(X), \pi_k(Y)].$$

118. Let X be an $(n-1)$-connected CW-complex. Describe the homotopy group $\pi_n(X)$.

119. Compute the homotopy group $\pi_3(S^2 \vee S^2)$.

120. Define when a map $f : X \longrightarrow Y$ is a weak homotopy equivalence. Outline the proof that the following two statements are equivalent

(1) The map $f : X \longrightarrow Y$ is weak homotopy equivalence.

(2) The induced homomorphism $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is isomorphism for all n and $x_0 \in X$.

121. Let X, Y be CW-complexes. Prove that if a map $f_* : X \longrightarrow Y$ induces isomorphism

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$$

for all $n \geq 0$ and $x_0 \in X$, then f is a homotopy equivalence.

122. Let X be a Hausdorff topological space. Prove that there exists a CW-complex K and a weak homotopy equivalence $f : K \longrightarrow X$. Show that the CW-complex K is unique up to homotopy equivalence.

123. Let X, Y be two weak homotopy equivalent spaces. Prove that there exist a CW-complex K and maps $f : K \longrightarrow X$, $g : K \longrightarrow Y$ which weak homotopy equivalences.

124. Define an Eilenberg-McLane space. Prove that it does exists and unique up to weak homotopy equivalence.

125. Construct the space $K(\pi, 1)$, where π is a finitely generated abelian group.

126. Let $X = K(\pi, n)$. Prove that $\Omega X = K(\pi, n-1)$.

127. Let X be a CW-complex, and $n \geq 1$. Construct a CW-complex X_n and a map $f_n : X \longrightarrow X_n$ such that

(1) $\pi_q(X_n) = \begin{cases} \pi_q(X) & \text{if } q \leq n \\ 0 & \text{else} \end{cases}$

(2) $(f_n)_* : \pi_q(X) \longrightarrow \pi_q(X_n)$ is isomorphism if $q \leq n$.

128. Let X be a CW-complex, and $n \geq 1$. Construct a CW-complex $X|_n$ and a map $g_n : X|_n \longrightarrow X$ such that

(1) $\pi_q(X|_n) = \begin{cases} \pi_q(X) & \text{if } q \geq n \\ 0 & \text{else} \end{cases}$

(2) $(g_n)_* : \pi_q(X|_n) \longrightarrow \pi_q(X)$ is isomorphism if $q \geq n$.

129. Let $X = S^2$. Prove that $X|_3 = S^3$.

130. Let $X = \mathbf{CP}^n$. Prove that $X|_3 = X|_{2n+1} = S^{2n+1}$.

131. Define the complex $\mathcal{C}(X)$ and the homology groups $H_q(X)$. Calculate the homology groups for $X = \{pt\}$.

132. Define chain maps and chain homotopy. Prove that two chain homotopic maps $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ induce the same homomorphism in homology groups.

133. Let $g, h : X \to Y$ be homotopic maps. Prove that $g_* = h_* : H_q(X) \to H_q(Y)$.

134. Let X and Y be homotopy equivalent spaces. Prove that then $H_q(X) \cong H_q(Y)$ for all q.

135. Prove that $H_0(X) \cong \mathbf{Z}$ if X is a path-connected space.

136. Prove that if $f : X \to Y$ is a map of path-connected spaces, then $f_* : H_0(X) \to H_0(Y)$ is an isomorphism.

137. Define relative homology groups. State and prove the LES-Lemma.

138. Let $B \subset A \subset X$ be a triple of spaces. Prove that there is a long exact sequence in homology:

$$\cdots \to H_q(A, B) \xrightarrow{i_*} H_q(X, B) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A, B) \xrightarrow{i_*} \cdots .$$

139. Let (X, A) be a pair of spaces. Prove that the inclusion $i : (X, A) \to (X \cup C(A), C(A))$ induces the isomorphism $H_q(X, A) \cong H_q(X \cup C(A), C(A)) = H_q(X \cup C(A), v)$.

140. Define the operation $\beta : \mathcal{C}(X) \to \mathcal{C}(X)$ (induced by the barycentric subdivision). Prove that the chain map $\beta : \mathcal{C}(X) \to \mathcal{C}(X)$ induces the identity homomorphism in homology:

$$Id = \beta_* : H_q(\mathcal{C}(X)) \to H_q(\mathcal{C}(X)) \quad \text{for each } q \geq 0.$$

141. Define the chain complex $\mathcal{C}^{\mathbf{U}}(X)$ for a covering $\mathbf{U}$. Prove that the inclusion $\mathcal{C}^{\mathbf{U}}(X) \subset \mathcal{C}(X)$ induces an isomorphism in homology groups.

142. State and prove the Excision Theorem.

143. Let $X = X_1 \cup X_2$. Prove that the following sequence of complexes is exact

$$0 \to \mathcal{C}(X_1 \cap X_2) \xrightarrow{\alpha} \mathcal{C}(X_1) \oplus \mathcal{C}(X_2) \xrightarrow{\beta} \mathcal{C}(X_1) + \mathcal{C}(X_2) \to 0.$$

144. Let $X_1, X_2 \subset X$, and $X_1 \cup X_2 = X$, $\overset{o}{X}_1 \cup \overset{o}{X}_2 = X$. Prove that the chain map

$$\mathcal{C}(X_1) + \mathcal{C}(X_2) \to \mathcal{C}(X_1 \cup X_2)$$

induces isomorphism in the homology groups.

145. State and prove the Mayer-Vietoris Theorem.

146. Compute the homology groups $H_q(S^n)$.

147. Let X be a space. Prove that $\widetilde{H}_{q+1}(\Sigma X) \cong \widetilde{H}_q(X)$ for each q.

148. Let A be a set of indices, and S^n_α be a copy of the n-th sphere, $\alpha \in A$. Compute the homology groups $\widetilde{H}_q\left(\bigvee_{\alpha\in A} S^n_\alpha\right)$.

149. Let (X_α, x_α) be based spaces, $\alpha \in A$. Assume that the pair (X_α, x_α) is Borsuk pair for each $\alpha \in A$. Prove that

$$\widetilde{H}_q\left(\bigvee_{\alpha\in A} X_\alpha\right) = \bigoplus_{\alpha\in A} \widetilde{H}_q(X_\alpha).$$

150. Let $f : S^n \to S^n$ be a map of degree $d = \deg f$. Prove that $f_* : H_n(S^n) \to H_n(S^n)$ is a multiplication by d.

151. Let $g : \bigvee_{\alpha\in A} S^n_\alpha \xrightarrow{g} \bigvee_{\beta\in B} S^n_\beta$ be a map. Prove that the homomorphism

$$\bigoplus_{\alpha\in A} \mathbf{Z}(\alpha) = H_n\left(\bigvee_{\alpha\in A} S^n_\alpha\right) \xrightarrow{g_*} H_n\left(\bigvee_{\beta\in B} S^n_\beta\right) = \bigoplus_{\beta\in B} \mathbf{Z}(\beta)$$

is given by multiplication with matrix $\{d_{\alpha\beta}\}_{\alpha\in A,\beta\in B}$, where $d_{\alpha\beta} = \deg g_{\alpha\beta}$. (Define the maps $g_{\alpha\beta}$.)

152. Define the cellular chain complex $\mathcal{E}(X)$. Prove that the following composition is trivial

$$\mathcal{E}_{q+1}(X) \xrightarrow{\flat\partial_{q+1}} \mathcal{E}_q(X) \xrightarrow{\flat\partial_q} \mathcal{E}_{q-1}(X).$$

153. Prove that there is an isomorphism $H_q(\mathcal{E}(X)) \cong H_q(X)$ for each q and any CW-complex X.

154. Let X be a CW-complex, and e^q be a q-cell and σ^{q-1} be a $(q-1)$-cell of X. Define the incidence coefficient $[e^q : \sigma^{q-1}]$. Prove that the boundary operator $\flat\partial_q : \mathcal{E}_q(X) \to \mathcal{E}_{q-1}(X)$ is given by the formula:

$$\flat\partial_q(e^q) = \sum_{j\in E_{q-1}} [e^q : \sigma^{q-1}_j]\sigma^{q-1}_j.$$

155. Let $A : S^n \to S^n$ be the antipodal map, $A : x \mapsto -x$, and $\iota_n \in \pi_n(S^n)$ be the generator represented by the identity map $S^n \to S^n$. Prove that the homotopy class $[A] \in \pi_n(S^n)$ is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

156. Let $e^0, \ldots, e^n$ be the cells in the standard cell decomposition of $\mathbf{RP}^n$. Prove that

$$[e^q : e^{q-1}] = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 0, & \text{if } q \text{ is even.} \end{cases}$$

157. Compute the homology groups $H_q(\mathbf{RP}^n)$, $H_q(\mathbf{CP}^n)$.

158. Compute the homology groups $H_q(\mathbf{RP}^{2n} \# \mathbf{CP}^n)$.

159. Prove that there is no map $f : D^n \to S^{n-1}$ so that the restriction $f|_{S^{n-1}} : S^{n-1} \to S^{n-1}$ has nonzero degree.

160. Let X be a topological space, $\alpha \in H_q(X)$. Prove that there exist a CW-complex K, a map $f : K \to X$, an element $\beta \in H_q(K)$ such that $f_*(\beta) = \alpha$.

161. Let $f : X \to Y$ be a weak homotopy equivalence. Prove that the homomorphism $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for all $q \geq 0$.

162. Show that the spaces $\mathbf{CP}^\infty \times S^3$ and S^2 have isomorphic homotopy groups and that they are not homotopy equivalent.

163. Show that the spaces $\mathbf{RP}^n \times S^m$ and $S^n \times \mathbf{RP}^m$ $(n \neq m)$ have isomorphic homotopy groups and they are not homotopy equivalent.

164. Show that the spaces $S^1 \vee S^1 \vee S^2$ and $S^1 \times S^1$ have the same homology groups and different homotopy groups.

165. Show that the projection

$$S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1)/(S^1 \vee S^1) = S^2$$

induces trivial homomorphism in homotopy groups.

166. Define the Hurewicz homomorphism $h : \pi_n(X, x_0) \to H_n(X)$. Prove that h is a homomorphism.

167. Let $x_0, x_1 \in X$, and $\gamma : I \to X$ be a path connecting the points x_0, x_1: $\gamma(0) = x_0$, and $\gamma(1) = x_1$. The path γ determines the isomorphism $\gamma_\# : \pi_n(X, x_0) \to \pi_n(X, x_1)$. Prove that the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\gamma_\#} & \pi_n(X, x_1) \\ & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle h} & \\ & H_n(X) & \end{array}$$

168. (Hurewicz Theorem) Let (X, x_0) be a based space, such that

$$\pi_0(X, x_0) = 0, \ \pi_1(X, x_0) = 0, \cdots, \pi_{n-1}(X, x_0) = 0, \qquad (2)$$

where $n \geq 2$. Prove that

$$H_1(X) = 0, \quad H_2(X) = 0, \cdots, H_{n-1}(X) = 0,$$

and the Hurewicz homomorphism $h : \pi_{n-1}(X, x_0) \to H_n(X)$ is an isomorphism.

169. Let X be a simply-connected CW-complex with $\widetilde{H}_n(X) = 0$ for all n. Prove that X is contractible.

170. Let X be a simply connected space, and $H_1(X) = 0$, $H_2(X) = 0 \ \cdots \ H_{n-1}(X) = 0$. Prove that $\pi_1(X) = 0$, $\pi_2(X) = 0 \ \cdots \ \pi_{n-1}(X) = 0$ and the Hurewicz homomorphism $h : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.

171. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{proj}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that g induces trivial homomorphism in homology and homotopy groups, however g is not homotopic to a constant map.

172. Let X be a connected space. Prove that the Hurewicz homomorphism $h : \pi_1(X, x_0) \to H_1(X)$ is epimorphism, and the kernel of h is the commutator $[\pi_1(X, x_0), \pi_1(X, x_0)] \subset \pi_1(X, x_0)$.

173. State the relative version of the Hurewicz Theorem. State and prove the Whitehead Theorem-II. Let X, Y be simply connected spaces and $f : X \to Y$ be a map which induces isomorphism $f_* : H_q(X) \to H_q(Y)$ for all $q \geq 0$. Prove that f is weak homotopy equivalence.

174. Define homology and cohomology groups with coefficients in an abelian group G. Compute the groups $H_q(\mathbf{RP}^n; \mathbf{Z}/p)$, $H^q(\mathbf{RP}^n; \mathbf{Z}/p)$ for any prime p.

175. Consider the short exact sequence $0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to \mathbf{Z}/2 \to 0$. Compute the connecting homomorphisms

$$\partial = \beta^m : H^q(\mathbf{RP}^n; \mathbf{Z}/2) \to H^{q+1}(\mathbf{RP}^n; \mathbf{Z})$$

176.* Let G be an abelian group, $0 \to R \xrightarrow{\beta} F \xrightarrow{\alpha} G \to 0$, be a free resolution of G, and H be an arbitrary abelian group. Prove that the sequence

$$0 \to \mathrm{Ker}(\beta \otimes 1) \to R \otimes H \xrightarrow{\beta \otimes 1} F \otimes H \xrightarrow{\alpha \otimes 1} G \otimes H \to 0$$

is exact.

177.* Prove that the group $\mathrm{Tor}(G, H)$ is well-defined, i.e. it does not depend on the choice of resolution.

178.* Let G, H be abelian groups. Prove that there is a canonical isomorphism $\mathrm{Tor}(G, H) \cong \mathrm{Tor}(H, G)$.

179. Let F be a free abelian group. Show that $\mathrm{Tor}(F, G) = 0$ for any abelian group G.

180. Let G be an abelian group. Denote $T(G)$ a maximal torsion subgroup of G. Show that $\mathrm{Tor}(G, H) \cong T(G) \otimes T(H)$ for finite generated abelian groups G, H. Give an example of abelian groups G, H, so that $\mathrm{Tor}(G, H) \neq T(G) \otimes T(H)$.

181. Let X be a space, G be an abelian group. Prove that there is a split short exact sequence

$$0 \to H_q(X) \otimes G \to H_q(X; G) \to \mathrm{Tor}(H_{q-1}(X), G) \to 0$$

182*. Let G be an abelian group, $0 \to R \xrightarrow{\beta} F \xrightarrow{\alpha} G \to 0$ be a free resolution, and let H be an abelian group. Prove that the following sequence is exact:

$$0 \longleftarrow \mathrm{Coker}\ \beta^{\#} \longleftarrow \mathrm{Hom}(R, H) \xleftarrow{\beta^{\#}} \mathrm{Hom}(F, H) \xleftarrow{\alpha^{\#}} \mathrm{Hom}(G, H) \longleftarrow 0.$$

183*. Prove that the group $\mathrm{Ext}(G, H)$ is well defined, i.e. it does not depend on the choice of free resolution of G.

184*. Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of abelian groups. Prove that it induces the following exact sequence:

$$0 \to \mathrm{Hom}(G'', H) \to \mathrm{Hom}(G, H) \to \mathrm{Hom}(G', H) \to$$
$$\mathrm{Ext}(G'', H) \to \mathrm{Ext}(G, H) \to \mathrm{Ext}(G', H) \to 0$$

185. Prove that $\mathrm{Ext}(\mathbf{Z}, H) = 0$ for any group H.

186. Prove the isomorphisms: $\mathrm{Ext}(\mathbf{Z}/m, \mathbf{Z}/n) \cong \mathbf{Z}/m \otimes \mathbf{Z}/n$, $\mathrm{Ext}(\mathbf{Z}/m, \mathbf{Z}) \cong \mathbf{Z}/m$.

187. Let X be a space, G an abelian group. Prove that there is a split exact sequence

$$0 \to \mathrm{Ext}(H_{q-1}(X), G) \to H^q(X; G) \to \mathrm{Hom}(H_q(X), G) \to 0 \tag{3}$$

for each $q \geq 0$.

188. Let X be a space, and G an abelian group. Prove that there is a split exact sequence

$$0 \to H^q(X; \mathbf{Z}) \otimes G \to H^q(X; G) \to \mathrm{Tor}(H^{q+1}(X; \mathbf{Z}), G) \to 0$$

for any $q \geq 0$.

189. Let G be a finitely generated abelian group. Let $F(G)$ be the maximum free abelian subgroup of G, and $T(G)$ be the maximum torsion subgroup. Let X be a space such that the groups $H_q(X)$ are finitely generated for all q. Prove that $H^q(X; \mathbf{Z})$ are also finitely generated and $H^q(X; \mathbf{Z}) \cong F(H_q(X; \mathbf{Z})) \oplus T(H_{q-1}(X; \mathbf{Z}))$.

190. Let F be $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$. Prove that

$$H_q(X; F) = H_q(X) \otimes F, \quad H^q(X; F) = \mathrm{Hom}(H_q(X), F).$$

191. Let X be a finite CW-complex, and $\mathbf{F}$ be a field. Prove that the number

$$\chi(X)_{\mathbf{F}} = \sum_{q \geq 0} (-1)^q \dim\ H_q(X; \mathbf{F})$$

does not depend on the field $\mathbf{F}$ and is equal to the Euler characteristic

$$\chi(X) = \sum_{q \geq 0} (-1)^q \{\# \text{ of } q\text{-cells of } X\}.$$

192. Let a finite CW-complex X be a union of two CW-subcomplexes: $X = X_1 \cup X_2$, where $X_1 \cap X_2 \subset X$ is a CW-subcomplex as well. Prove that

$$\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

193. Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes. Define the tensor product $\bar{\mathcal{C}}_* = \mathcal{C}_* \otimes \mathcal{C}'_*$. Prove that $\bar{\partial}_{q+1}\bar{\partial}_q = 0$.

194. Let $\mathcal{E}_* = \mathcal{E}_*(X)$, $\mathcal{E}'_* = \mathcal{E}_*(X')$. Define the complexes $\mathcal{E}_*(r)$, $\mathcal{E}'_*(s)$ and compute the homology groups of the tensor product of these chain complexes: $\mathcal{E}_*(r) \otimes \mathcal{E}'_*(s)$.

195. Use the result of # 67 to prove the Künneth formula for homology groups:

$$0 \to \bigoplus_{r+s=q} H_r(X) \otimes H_s(X') \to H_q(X \times X') \to \bigoplus_{r+s=q-1} \mathrm{Tor}(H_r(X), H_s(X')) \to 0$$

196. Outline the proof of the Künneth formula for cohomology groups:

$$0 \to \bigoplus_{r+s=q} H^r(X) \otimes H^s(X') \to H^q(X \times X') \to \bigoplus_{r+s=q+1} \mathrm{Tor}(H^r(X), H^s(X')) \to 0.$$

197. Let F be a field. Prove that

$$H_q(X \times X'; F) \cong \bigoplus_{r+s=q} H_r(X; F) \otimes H_s(X'; F),$$

$$H^q(X \times X'; F) \cong \bigoplus_{r+s=q} H^r(X; F) \otimes H^s(X'; F).$$

198. Let $\beta_q(X) = \text{Rank} H_q(X)$ be the Betti number of X. Prove that

$$\beta_q(X \times X') = \sum_{r+s=q} \beta_r(X)\beta_s(X').$$

199. Let X, X' be such spaces that their Euler characteristics $\chi(X)$, $\chi(X')$ are finite. Prove that $\chi(X \times X') = \chi(X) \cdot \chi(X')$.

200. Prove the Lefschetz Theorem: Let X be a finite CW-complex, $f : X \to X$ be a map such that $\text{Lef}(f) = 0$. Then f has a fixed point, i.e. such point $x_0 \in X$ that $f(x_0) = x_0$.

201. Let X be a finite contractible CW-complex. Prove that any map $f : X \to X$ has a fixed point.

202. Define a flow of homeomorphisms $\phi_t : X \to X$. Let X be a finite CW-complex with $\chi(X) \neq 0$, and $\phi_t : X \to X$ be a flow. Prove that there exists a point $x_0 \in X$ so that $\phi_t(x_0) = x_0$ for all $t \in \mathbf{R}$.

203. Let $f : \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ be a map. Prove that f always has a fixed point. Give an example that the above statement fails for a map $f : \mathbf{RP}^{2n+1} \to \mathbf{RP}^{2n+1}$.

204. Let $n \neq k$. Prove that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^k$.

205. Let $f : S^n \to S^n$ be a map, and $\deg(f)$ be the degree of f. Prove that $\text{Lef}(f) = 1 + (-1)^n \deg(f)$.

206. Prove that there is no tangent vector field $v(x)$ on the sphere S^{2n} such that $v(x) \neq 0$ for all $x \in S^{2n}$. Construct everywhere non-zero vector field v on S^{2n+1}.

207. Let $K \subset S^n$ be homeomorphic to the cube I^k, $0 \leq k \leq n$. Prove that $\widetilde{H}_q(S^n \setminus K) = 0$ for all $q \geq 0$.

208. Let $S^k \subset S^n$, $0 \leq k \leq n-1$. Prove that

$$\widetilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbf{Z}, & \text{if } q = n-k-1, \\ 0 & \text{if } q \neq n-k-1. \end{cases}$$

209. State and prove the Jordan-Brouwer Theorem.

210. State and prove the Brouwer Invariance Domain Theorem.

211. Let (X, A) be a CW-pair. Prove that the group $H^1(X, A; \mathbf{Z})$ is a free abelian group.

212. Define the cup-product in cohomology. Prove that $\delta(\phi \cup \psi) = (\delta\phi) \cup \psi + (-1)^k \phi \cup (\delta\psi)$ where $\phi \in C^k(X)$, $\psi \in C^l(X)$.

213. Compute the cup product of $H^*(\mathbf{RP}^2; \mathbf{Z}/2)$, $H^*(M_g^2; \mathbf{Z})$.

214. Prove that $\alpha\beta = (-1)^{kl}\beta\alpha$ if $\alpha \in H^k(X)$, $\beta \in H^l(X)$.

215. Define the external product

$$\mu : H^*(X; R) \otimes H^*(Y; R) \to H^*(X \times Y; R).$$

Define the ring structure on $H^*(X; R) \otimes H^*(Y; R)$. Prove that the external product $\mu : H^*(X; R) \otimes H^*(Y; R) \to H^*(X \times Y; R)$ induces a ring isomorphism provided that $H^q(Y; R)$ are free R-modules for all q.

216. Let $\Delta : X \to X \times X$ be a diagonal map. Prove that the homomorphism

$$H^k(X; R) \otimes H^l(X; R) \xrightarrow{\mu} H^{k+l}(X \times X; R) \xrightarrow{\Delta^*} H^{k+l}(X; R)$$

coincides with the cup-product, i.e. that $\Delta^*(\mu(\alpha \otimes \beta)) = \alpha \cup \beta$.

217. State the Poincarè Duality Theorem. Compute the Poincarè Duality for M_g^2.

218. Prove that the odd-dimensional manifold has zero Euler characteristic.

219. Prove that $\langle \alpha \cup \beta, \mu \rangle = \langle \beta, \mu \cap \alpha \rangle$.

220. Let M^{4k} be a compact oriented manifold, and $V = H^{2k}(M^{4k};\mathbf{Z})/\mathrm{Tor}$. Use the Poincarè duality to prove that the pairing

$$\mu(\alpha,\beta) = \langle \alpha \cup \beta, [M^{4k}] \rangle$$

defines a nondegenerated quadratic form on V. Compute the index of this quadratic form for $\mathbf{CP}^{2n}$.

221. Use Poincaré duality to prove that $H^*(\mathbf{CP}^n;\mathbf{Z}) \cong \mathbf{Z}[x]/x^{n+1}$.

222. Use Poincaré duality to prove that $H^*(\mathbf{RP}^n;\mathbf{Z}/2) \cong \mathbf{Z}/2[x]/x^{n+1}$.

223. Let $f : \mathbf{CP}^{2n} \to \mathbf{CP}^{2n}$ be a map. Show that f has a fixed point.

224. Compute the ring structure $H^*(\mathbf{RP}^n;\mathbf{Z}/2^k)$.

225. Let $n > k$. Prove that there is no map $f : \mathbf{RP}^n \to \mathbf{RP}^k$ which induces a nontrivial ring homomorphism $f^* : H^*(\mathbf{RP}^k;\mathbf{Z}/2) \to H^*(\mathbf{RP}^n;\mathbf{Z}/2)$.

226. Let a map $h : \mathbf{RP}^{n-1} \times \mathbf{RP}^{n-1} \to \mathbf{RP}^{n-1}$, be such that the induced homomorphism

$$h^* : H^*(\mathbf{RP}^{n-1};\mathbf{Z}/2) \to H^*(\mathbf{RP}^{n-1} \times \mathbf{RP}^{n-1} : \mathbf{Z}/2)$$

takes generator $y \in H^1(\mathbf{RP}^{n-1};\mathbf{Z}/2)$ to the sum of generators: $h^*(y) = x_1 \otimes 1 + 1 \otimes x_2$. Prove that n must be a power of 2.

227. Prove that $\mathbf{RP}^3$ and is not homotopy equivalent to $S^3 \vee \mathbf{RP}^2$.

228. Recall that $\mathbf{RP}^{2n+1}$ is a factor-space of S^{2n+1} by the free action of $\mathbf{Z}/2$ (antipodal map gives a generator). The complex projective space $\mathbf{CP}^n$ is the factor-space of S^{2n+1} by the action of the circle $S^1 = \{e^{i\phi}\}$. Notice that the group $\mathbf{Z}/2$ is a subgroup of the circle. Thus the identity map $i : S^{2n+1} \to S^{2n+1}$ determimes the map $g : \mathbf{RP}^{2n+1} \to \mathbf{CP}^n$. Compute the ring homomorphism $g^* : H^*(\mathbf{CP}^n;\mathbf{Z}/2) \to H^*(\mathbf{RP}^{2n+1};\mathbf{Z}/2)$.

229. Let X be a space, and F be a field. The *Poincare series* of X is the formal series

$$p_X(t) = \sum_{j=0}^{\infty} \dim_F H^i(X;F).$$

Prove that $p_{X\times Y}(t) = p_X(t)p_Y(t)$. Compute the Poincare series for the spaces:

$$\mathbf{CP}^{n_1} \times \cdots \mathbf{CP}^{n_k}, \quad \mathbf{RP}^{m_1} \times \cdots \mathbf{RP}^{m_\ell}, \quad \mathbf{CP}^{n_1} \times \cdots \mathbf{CP}^{n_k} \times \mathbf{RP}^{m_1} \times \cdots \mathbf{RP}^{m_\ell}.$$

230. Define the Hopf invariant $h(\lambda)$ of an element $\lambda \in \pi_{4q-1}(S^{2q})$.

231. Prove that $h(\lambda_1 + \lambda_2) = h(\lambda_1) + h(\lambda_2)$.

232. Prove that there is an element in $\pi_{4n-1}(S^{2n})$ with the Hopf invariant 2. State and prove the theorem that the group $\pi_{4n-1}(S^{2n})$ is infinite.

233. Prove that $h([\iota_{2q},\iota_{2q}]) = 2$, where $\iota_{2q} \in \pi_{2q}(S^{2q})$ is the standard generator.

234. Define a *cohomology operation*. Give examples.

235. Define a canonical *fundamental* class

$$\iota_n \in \mathrm{Hom}(H_n(K(\pi,n);\mathbf{Z}),\pi).$$

236. Let π, π' be abelian groups. Prove that there is a bijection

$$[K(\pi,n),K(\pi',n)] \leftrightarrow \mathrm{Hom}(\pi,\pi').$$

237. Let π be an abelian group and n be a positive integer. Prove that the homotopy type of the Eilenberg-McLane space $K(\pi, n)$ is completely determined by the group π and the integer n.

238. Prove that there is a bijection

$$\mathcal{O}(\pi, n; \pi', n') \leftrightarrow H^{n'}(K(\pi, n), \pi')$$

given by the formula $\theta \leftrightarrow \theta(\iota_n)$.

239. Let Y be a homotopy simple space, (B, A) a CW-pair and $X^n = B^{(n)} \cup A$ for $n = 0, 1, \ldots$. Define the obstruction cochain

$$c(f) \in \mathcal{E}^{n+1}(B, A; \pi_n(Y)) = \mathrm{Hom}(\mathcal{E}_{n+1}(B, A), \pi_n(Y)).$$

Prove that $c(f)$ is a cocycle.

240. Let Y be a homotopy simple space, (B, A) a CW-pair and $X^n = B^{(n)} \cup A$ for $n = 0, 1, \ldots$. Prove that a map $f : X^n \to Y$ can be extended to a map $\tilde{f} : X^{n+1} \to Y$ if and only if $c(f) = 0$.

241. Define $d(f, g) \in \mathcal{E}^n(B, A; \pi_n(Y))$. Prove the formula: $\delta d(f, g) = c(g) - c(f)$.

242. Let Y be a homotopy simple space, (B, A) a CW-pair and $X^n = B^{(n)} \cup A$ for $n = 0, 1, \ldots$. Let $f : X^n \to Y$ be a map, and $d \in \mathcal{E}^n(B, A; \pi_n(Y))$ is a cochain. Prove that there exists a map $g : X^n \to Y$ such that $f|_{X^{n-1}} = g|_{X^{n-1}}$ and $d(f, g) = d$.

243. Let Y be a homotopy simple space, (B, A) a CW-pair and $X^n = B^{(n)} \cup A$ for $n = 0, 1, \ldots$. Assume $f : X^n \to Y$ is a map. Prove that there exists a map $g : X^{n+1} \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ if and only if $[c(f)] = 0$ in $H^{n+1}(B, A; \pi_n Y)$.

244. Prove the following result
Theorem. Let $f, g : K \to Y$ be two maps, where K is a CW-complex and Y is homotopy-simple space. Assume that $f|_{K^{(n-1)}} = g|_{K^{(n-1)}}$. Then the cohomology class $[d(f, g)] \in H^n(K, \pi_n Y)$ vanishes if and only if there exists a homotopy between the maps $f|_{K^{(n)}}$ and $g|_{K^{(n)}}$ relative to the skeleton $K^{(n-2)}$.

245. Prove the following result:
Theorem. There is a bijection

$$[X, K(\pi, n)] \leftrightarrow H^n(X; \pi).$$

given by the formula $[f] \mapsto f^* \iota_n$.

246. Consider a k-torus T^k. We identify T^k with the quotient space $\mathbf{R}^k / \sim$, where two vectors $\vec{x} \sim \vec{y}$ if and only if all coordinates of the vector $\vec{x} - \vec{y}$ are integers. It is easy to see that a linear map $\bar{f} : \mathbf{R}^k \to \mathbf{R}^\ell$ given by an $k \times \ell$-matrix A with integral entries descends to a map $f : T^k \to T^\ell$. Prove that any map $g : T^k \to T^\ell$ is homotopic to a linear map as above.

247. Prove the following result:
Theorem. Let X be an n-dimensional CW-complex. Then there is a bijection:

$$H^n(X; \mathbf{Z}) \cong [X, S^n].$$