

### QUESTIONS FOR THE MIDTERM, SPRING 2020

1. Basic spaces:  $\mathbf{R}^n$ ,  $S^n$ , stereographic projection. The space  $S^\infty$ .
2. Projective spaces  $\mathbf{RP}^n$ ,  $\mathbf{CP}^n$ ,  $\mathbf{HP}^n$ : definitions, local coordinate system, the Hopf maps  $S^n \rightarrow \mathbf{RP}^n$ ,  $S^{2n+1} \rightarrow \mathbf{CP}^n$ ,  $S^{4n+3} \rightarrow \mathbf{HP}^n$ .
3. Prove the homeomorphisms:  $\mathbf{RP}^1 \cong S^1$ ,  $\mathbf{CP}^1 \cong S^2$ ,  $\mathbf{HP}^1 \cong S^4$ .
4. Prove that  $\mathbf{RP}^n$ ,  $\mathbf{CP}^n$ ,  $\mathbf{HP}^n$  are connected and compact spaces.
5. Define Grassmannian manifolds  $G_k(\mathbf{R}^n)$ ,  $G_k(\mathbf{C}^n)$ : and construct local coordinate systems, in particular, find their dimensions.
6. Prove that the Grassmannian manifolds  $G_k(\mathbf{R}^n)$ , and  $G_k(\mathbf{C}^n)$  are compact and connected.
7. Define classic Lie groups  $GL(\mathbf{R}^k)$ ,  $GL(\mathbf{C}^k)$ ,  $O(k)$ ,  $SO(k)$ ,  $U(k)$ ,  $SU(k)$ . Prove that the spaces  $O(n)$ ,  $SO(n)$ ,  $U(n)$ ,  $SU(n)$  are compact. How many connected components does each of these spaces have?
8. Prove that  $SO(2)$  and  $U(1)$  are homeomorphic to  $S^1$ ,  $SO(3)$  is homeomorphic to  $\mathbf{RP}^3$ , and  $SU(2)$  is homeomorphic to  $S^3$ .
9. Prove that  $SO(4) \cong SO(3) \times S^3$ .
10. Define Stiefel manifolds  $V_k(\mathbf{R}^n)$ ,  $V_k(\mathbf{C}^n)$ ,  $V_k(\mathbf{H}^n)$ . Prove the following homeomorphisms:

$$V_n(\mathbf{R}^n) \cong O(n), \quad V_{n-1}(\mathbf{R}^n) \cong SO(n),$$

$$V_n(\mathbf{C}^n) \cong U(n), \quad V_{n-1}(\mathbf{C}^n) \cong SU(n),$$

$$V_1(\mathbf{R}^n) \cong S^{n-1}, \quad V_1(\mathbf{C}^n) \cong S^{2n-1}, \quad V_1(\mathbf{H}^n) \cong S^{4n-1}.$$

11. Define action of the groups  $O(k)$ ,  $U(k)$  on the Stiefel manifolds  $V_k(\mathbf{R}^n)$ ,  $V_k(\mathbf{C}^n)$ . Prove the following homeomorphisms:  $V_k(\mathbf{R}^n)/O(k) \cong G_k(\mathbf{R}^n)$ ,  $V_k(\mathbf{C}^n)/U(k) \cong G_k(\mathbf{C}^n)$ .
12. Prove the following homeomorphisms:

$$S^{n-1} \cong O(n)/O(n-1) \cong SO(n)/SO(n-1),$$

$$S^{2n-1} \cong U(n)/U(n-1) \cong SU(n)/SU(n-1),$$

$$G_k(\mathbf{R}^n) \cong O(n)/O(k) \times O(n-k), \quad G_k(\mathbf{C}^n) \cong U(n)/U(k) \times U(n-k).$$

13. Prove that the Klein bottle  $Kl^2$  is homeomorphic to the union of two Möbius bands along the circle.
14. Prove that  $Kl^2 \# \mathbf{RP}^2$  is homeomorphic to  $\mathbf{RP}^2 \# T^2$ .
15. Define a cylinder and a cone of a map  $f : X \rightarrow Y$ . Prove that the cones of the maps  $c : S^n \rightarrow \mathbf{RP}^n$  and  $h : S^{2n+1} \rightarrow \mathbf{CP}^n$  are homeomorphic to  $\mathbf{RP}^{n+1}$  and  $\mathbf{CP}^{n+1}$  respectively.
16. Define suspension. Prove that  $\Sigma(S^n) \cong S^{n+1}$ .
17. Define a compact-open topology on  $\mathcal{C}(X, Y)$ . Prove the homeomorphism:  $\mathcal{C}(X, \mathcal{C}(Y, Z)) \cong \mathcal{C}(X \times Y, Z)$  for Hausdorff and locally compact spaces  $X$ ,  $Y$ ,  $Z$ . Prove that this homeomorphism is natural.

18. Define the spaces of paths  $\mathcal{E}(X, x_0, x_1)$ ,  $\mathcal{E}(X, x_0)$ , and loops  $\Omega(X, x_0)$ . Prove that the spaces  $\Omega(S^n, x_0)$  and  $\Omega(S^n, x_1)$  are homeomorphic for any points  $x_0, x_1 \in S^n$ .
19. Let  $X, Y$  be pointed spaces. Prove the homeomorphism  $\mathcal{C}(\Sigma(X), Y) \cong \mathcal{C}(X, \Omega(Y))$  for Hausdorff and locally compact spaces  $X, Y$ . Prove that this homeomorphism is natural.
20. Define smash-product  $X \wedge Y$ . Prove that  $S^n \wedge S^k \cong S^{n+k}$  (as pointed spaces).
21. Define homotopy of two maps. Prove that the maps  $\phi^* : [X', Y] \rightarrow [X, Y]$ ,  $\psi_* : [X, Y] \rightarrow [X, Y']$  induced by maps  $\phi : X \rightarrow X'$ ,  $\psi : Y \rightarrow Y'$  are well-defined.
22. Give three definitions of homotopy equivalence. Prove that they are equivalent.
23. Prove that  $X \sim Y$  implies  $\Sigma(X) \sim \Sigma(Y)$  and  $\Omega(X) \sim \Omega(Y)$ .
24. Give a definition of a contractible space. Prove that  $\mathcal{E}(X, x_0)$  is a contractible.
25. Prove that a space  $X$  is contractible if and only if it is homotopy equivalent to a point.
26. Prove that a space  $X$  is contractible if and only if every map  $f : Y \rightarrow X$  is null-homotopic.
27. Give definition of a retract and deformational retract. Examples. Prove that  $\{0\} \cup \{1\}$  is not a retract of  $I = [0, 1]$ . Define map of pairs. Examples.
28. Define a  $CW$ -complex. Give examples of cell decomposition. Show that the axiom (W) does not imply the axiom (C) and vice-versa.
29. Construct a cellular decomposition of the wedge  $X = S^1 \vee S^2$  (with a single 2-cell  $e^2$ ) such that a closure of the cell  $e^2$  is not a  $CW$ -subcomplex of  $X$ .
30. Construct a cellular decomposition of the wedge  $X = \Sigma(S^n \vee S^k)$ . Prove that  $\Sigma(S^n \vee S^k) \sim S^{n+1} \vee S^{k+1}$ .
31. Prove that a  $CW$ -complex compact if and only if it is finite.
32. Construct a cellular decomposition of  $S^n, D^n, \mathbf{RP}^n, \mathbf{CP}^n, \mathbf{HP}^n$ .
33. Construct a cellular decomposition of the oriented 2-manifold of genus  $g$ .
34. Define the Schubert cells  $e(\sigma)$  corresponding to the Schubert symbol  $\sigma$ . Give examples.
35. Define the spaces  $H^j, \overline{H}^j$ . Prove that a  $k$ -plane  $\pi$  belongs to  $e(\sigma)$  if and only if there exists its basis  $v_1, \dots, v_k$ , such that  $v_1 \in H^{\sigma_1}, \dots, v_k \in H^{\sigma_k}$ .
36. Prove the following statement: Let  $\pi \in e(\sigma)$ , where  $\sigma = (\sigma_1, \dots, \sigma_n)$ . Then there exists a unique orthonormal basis  $v_1, \dots, v_k$  of  $\pi$ , so that  $v_1 \in H^{\sigma_1}, \dots, v_k \in H^{\sigma_k}$ .
37. Define the sets  $E(\sigma), \overline{E}(\sigma) \subset V_k(\mathbf{R}^n)$ . Prove that the set  $\overline{E}(\sigma) \subset V_k(\mathbf{R}^n)$  is homeomorphic to the closed cell of dimension  $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_k - k)$ . Furthermore the map  $q : e(\sigma) \rightarrow E(\sigma)$  is a homeomorphism.
38. Define the transformations  $T_{u,v}$ , prove its properties. Explain how the transformations  $T_{u,v}$  are used to prove that  $\overline{E}(\sigma) \subset V(n, k)$  is homeomorphic to a closed cell of dimension  $d(\sigma)$ .
39. Prove the statement: a collection of  $\binom{k}{n}$  Schubert cells  $e(\sigma)$  gives  $G_k(\mathbf{R}^n)$  a cell-decomposition.
40. Outline a construction of Schubert cells of the complex Grassmannian  $G_k(\mathbf{C}^n)$ .
41. Define when a pair  $(X, Y)$  is a Borsuk pair. Prove that a  $CW$ -pair  $(X, Y)$  is a Borsuk pair (in the case when  $X, Y$  are finite complexes).
42. Let  $(X, A)$  be a Borsuk pair. Prove that  $A$  is a deformation retract of  $X$  if and only if the inclusion  $A \rightarrow X$  is a homotopy equivalence.

43. Prove the statement: let  $X$  be a  $CW$ -complex and  $A \subset X$  be its contractible subcomplex. Then  $X$  is homotopy equivalent to the complex  $X/A$ .
44. Prove that for a  $CW$ -pair  $(X, A)$   $X/A \sim X \cup C(A)$ .
45. State Cellular Approximation Theorem. Prove it using Free Point Lemma.
46. State and prove Free Point Lemma.
47. Define homotopy groups  $\pi_n(X)$ . Prove that  $\pi_n(X)$  is commutative group for  $n \geq 2$ . Prove that  $\pi_k(S^n)$  is a trivial group for  $k < n$ .
48. Prove the statement: Let  $X$  be a  $CW$ -complex with only one zero-cell and without cells of dimension  $q < n$ , and  $Y$  be a  $CW$ -complex of dimension  $< q$ . Then any map  $Y \rightarrow X$  is homotopic to a constant map.
49. Define  $n$ -connected space. Prove the statement: Any  $n$ -connected  $CW$ -complex homotopy equivalent to a  $CW$ -complex with a single zero cell and without cells of dimensions  $1, 2, \dots, n$ .
50. Prove that if  $f, g : X \rightarrow Y$  are homotopic maps, then the homomorphisms  $f_*, g_* : \pi_n(X) \rightarrow \pi_n(Y)$  coincide.
51. Prove that if  $X$  is a path-connected space, then  $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ . Describe all isomorphisms here.
52. Prove that  $\pi_1 S^1 \cong \mathbf{Z}$ .
53. Prove that  $\pi_1(\bigvee_{\alpha \in A} S^1_\alpha)$  is a free group.
54. Prove that  $\pi_1(X, x_0) \cong \pi_1(X^{(2)}, x_0)$ , where  $X$  is a connected  $CW$ -complex and  $X^{(2)}$  its 2-skeleton.
55. Compute  $\pi_1(M^2)$  for two-dimensional oriented closed manifold of genus  $g$ , the sphere with  $g$  handles.
56. Compute  $\pi_1(M^2)$  for two-dimensional non-oriented closed manifold of genus  $g$ , the projective plane or the Klein bottle with  $g$  handles.
57. Let  $M = \mathbf{RP}^2 \# \dots \# \mathbf{RP}^2$  ( $n$  times). Compute  $\pi_1(M)$ .
58. Compute  $\pi_1(\mathbf{RP}^2 \# \mathbf{RP}^2)$  and  $\pi_1(Kl^2 \# \mathbf{RP}^2)$ .
59. Define  $G_1 * G_2$ . Give examples. Prove that  $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$ .
60. Define  $G_1 *_H G_2$ . Give examples. State and prove Van Kampen Theorem.
61. Define covering space. Give examples. Construct  $n$ -fold covering of  $S^1 \vee S^1$  (including  $n = \infty$ ).
62. State and prove Theorem on Covering Homotopy.
63. Prove that covering  $p : T \rightarrow X$  induces a monomorphism  $p_* : \pi_1(T, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ .
64. Prove that a loop  $\alpha_1 \cdots \alpha_k$ , where  $\alpha_j$  is a loop going along the  $j$ -th circle in the wedge  $\bigvee_{j=1}^k S^1_j$ , is not homotopic to zero.
65. Let  $p : T \rightarrow X$  be a covering, and  $f, g : Z \rightarrow T$  be two maps so that  $p \circ f = p \circ g$ , where  $Z$  is path-connected. Assume that  $f(z) = g(z)$  for some point  $z \in Z$ . Prove that  $f = g$ .
66. Prove that  $\pi_k(\mathbf{RP}^n) = 0$  if  $1 < k < n$ .
67. Prove that any map  $f : \mathbf{RP}^2 \rightarrow S^1$  is homotopic to a constant map.
68. Let  $Kl^2$  be the Klein bottle. Construct two-folded covering space  $T^2 \rightarrow Kl^2$ . Compute  $\pi_n(Kl^2)$  for all  $n$ .

69. Let  $p : T \rightarrow X$  be a covering,  $p(\tilde{x}_0) = x_0$ . Prove that there is one-to-one correspondence

$$\pi_1(X(X, x_0)/p_*(\pi_1(T, \tilde{x}_0))) \iff p^{-1}(x_0).$$

Prove that  $p^{-1}(x_0) \cong p^{-1}(x_1)$  for any points  $x_0, x_1 \in X$ .

70. Let  $p : T \rightarrow X$  be a covering map, and let  $\Gamma = p^{-1}(x_0)$ . Prove that  $\Gamma$  is a transitive right  $G$ -set for  $G = \pi_1(X, x_0)$ .
71. Let  $X$  be "good" space and  $G = \pi_1(X, x_0)$ . Prove that there is a bijection between isomorphism classes of covering spaces of  $X$  and transitive right  $G$ -sets given by

$$\{p : Y \rightarrow X\} \mapsto p^{-1}(x_0).$$

72. Let  $p : T \rightarrow X$  be a covering, and  $f : Z \rightarrow X$  be a map,  $f(z_0) = x_0$ , and  $\tilde{x}_0 \in T$  so that  $p(\tilde{x}_0) = x_0$  (here  $Z$  is path-connected space). Prove that there exists a lifting  $\tilde{f} : Z \rightarrow T$  of the map  $f$  so that  $\tilde{f}(z_0) = \tilde{x}_0$  if and only if  $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(T, \tilde{x}_0))$ .
73. Define morphism of two covering spaces  $T_1 p_1 \xrightarrow{p_1} X$  and  $T_2 \xrightarrow{p_2} X$ . Prove that two morphisms  $\phi, \phi' : T_1 \rightarrow T_2$  coincide if there is a point  $\tilde{x} \in T_1$  so that  $\phi(\tilde{x}) = \phi'(\tilde{x})$ .
74. Define a group of automorphisms (deck transformations)  $\text{Aut}(T \xrightarrow{p} X)$  of a covering  $p : T \rightarrow X$ . Prove that the group  $\text{Aut}(T \xrightarrow{p} X)$  acts on  $T$  without fixed points.
75. Let  $p : T \rightarrow X$  be a covering,  $p(\tilde{x}_0) = p(\tilde{x}'_0) = x_0$ , where  $\tilde{x}_0 \neq \tilde{x}'_0$ . Prove that there exists an automorphism  $\phi \in \text{Aut}(T \xrightarrow{p} X)$  such that  $\phi(\tilde{x}_0) = \tilde{x}'_0$  if and only if  $p_*(\pi_1(T, \tilde{x}_0)) = p_*(\pi_1(T, \tilde{x}'_0))$ .
76. Prove the following statement: Two covering spaces  $T_1 \xrightarrow{p_1} X$ ,  $T_2 \xrightarrow{p_2} X$  are isomorphic if and only if for any two points  $\tilde{x}_1, \tilde{x}_2 \in T$  such that  $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x$  the groups  $(p_1)_*(\pi_1(T_1, \tilde{x}_1))$ ,  $(p_2)_*(\pi_1(T_2, \tilde{x}_2))$  belong to the same conjugacy class in  $\pi_1(X, x)$ .
77. Let  $N(H)$  be a normalizer for a subgroup  $H$  of  $G$ . Prove the following statement: Let  $p : T \rightarrow X$  be a covering space. Then the group of automorphisms of this covering space is isomorphic to the group  $N(p_*(\pi_1(T, \tilde{x}_0)))/p_*(\pi_1(T, \tilde{x}_0))$ .
78. Define universal covering space over  $X$ . Prove the following statement: Let  $X$  be a path-connected CW-complex,  $x_0 \in X$ . Then for any subgroup  $G \subset \pi_1(X, x_0)$  there exists a covering  $T \xrightarrow{p} X$  and a point  $\tilde{x}_0 \in T$  so that  $p_*(\pi_1(T, \tilde{x}_0)) = G$ .
79. Define homotopy groups  $\pi_n(X, x_0)$ , in particular define the group operation and inverse. Prove that the groups  $\pi_n(X, x_0)$  are abelian if  $n \geq 2$ .
80. Prove that  $\pi_n(X \times Y, x_0 \times y_0) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$ . Compute  $\pi_n(T^k)$  for all  $n$ .
81. Let  $X$  be a path-connected space, and  $x_0, x_1 \in X$  be two different points. Let  $\gamma : I \rightarrow X$  be a path so that  $\gamma(0) = x_0$  and  $\gamma(1) = x_1$ . Define a homomorphism  $\gamma_{\#} : \pi_n(X, x_0) \rightarrow \pi_n(X, x_1)$ . Prove that  $\gamma_{\#}$  is an isomorphism.
82. Let  $M_g^2$  be a two-dimensional surface of genus  $g \geq 1$  (oriented). Compute the homotopy groups  $\pi_q(M_g^2)$ .
83. Define relative homotopy groups  $\pi_n(X, A; x_0)$ . Describe the group operation and the inverse element. Prove that the group  $\pi_n(X, A; x_0)$  is commutative for  $n \geq 3$ .
84. Define the homomorphisms in the following sequence:

$$\cdots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A; x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \cdots \quad (1)$$

Prove that the sequence (1) is exact.

85. Let  $A \subset X$  be a retract. Prove that
- $i_* : \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$  is monomorphism,
  - $j_* : \pi_n(X, x_0) \rightarrow \pi_n(X, A; x_0)$  is epimorphism,
  - $\partial : \pi_n(X, A; x_0) \rightarrow \pi_{n-1}(A, x_0)$  is zero homomorphism.
86. Let  $A$  be contractible in  $X$ . Prove that
- $i_* : \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$  is zero homomorphism,
  - $j_* : \pi_n(X, x_0) \rightarrow \pi_n(X, A; x_0)$  is monomorphism,
  - $\partial : \pi_n(X, A; x_0) \rightarrow \pi_{n-1}(A, x_0)$  is epimorphism.
87. State and prove Five-Lemma.
88. Let  $0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow 0$  be an exact sequence of finitely generated abelian groups. Prove that  $\sum_{i=1}^n (-1)^i \text{rank } A_i = 0$ .
89. Define locally trivial fiber bundle. Give several examples of non-trivial fiber bundles.
90. Prove that any locally-trivial fiber bundle over the cube  $I^q$  is trivial.
91. Define the covering homotopy property. Outline a proof that the covering homotopy property holds for a locally-trivial fiber bundle  $E \rightarrow B$ .
92. Define a Serre fiber bundle. Let  $Y$  be an arbitrary path-connected space,  $\mathcal{E}(Y, y_0)$  be the space of paths starting at  $y_0$ . Prove that the map  $p : \mathcal{E}(Y, y_0) \rightarrow Y$ , where  $p(s : I \rightarrow Y) = s(1) \in Y$  is a Serre fiber bundle.
93. Let  $A \subset X$ , and  $(X, A)$  be a Borsuk pair (for example, a  $CW$ -pair). Let  $E = \mathcal{C}(X, Y)$ ,  $B = \mathcal{C}(A, Y)$ , and the map  $p : E \rightarrow B$  be defined as  $p(f : X \rightarrow Y) = (f|_A : A \rightarrow Y)$ . Prove that the map  $p : E \rightarrow B$  is a Serre fiber bundle.
94. Define weak homotopy equivalence. Prove that finite  $CW$ -complexes  $X, Y$  are weak homotopy equivalent if and only if they are homotopy equivalent.
95. Let  $p : E \rightarrow B$  be Serre fiber bundle, where  $B$  be a path-connected space. Prove that the fibers  $F_0 = p^{-1}(x_0)$  and  $F_1 = p^{-1}(x_1)$  are weak homotopy equivalent for any two points  $x_0, x_1 \in B$ .
96. Prove that for any continuous map  $f : X \rightarrow Y$  there exists homotopy equivalent map  $f_1 : X_1 \rightarrow Y_1$ , such that  $f_1 : X_1 \rightarrow Y_1$  is Serre fiber bundle.
97. Let  $f : X \rightarrow Y$  be a continuous map. Prove that there exists a homotopy equivalent map  $g : X \rightarrow Y'$ , so that  $g$  is an inclusion.
98. Let  $p : E \rightarrow B$  be Serre fiber bundle,  $y \in E$  be any point,  $x = p(y)$ ,  $F = p^{-1}(x)$ . Prove that the homomorphism
- $$p_* : \pi_n(E, F; y) \rightarrow \pi_n(B, x)$$
- is an isomorphism for all  $n \geq 1$ .
99. Apply the homotopy exact sequence of Serre fibration to prove that (a)  $\pi_2(S^2) = \pi_1(S^1) = \mathbf{Z}$ ; (b)  $\pi_n(S^3) = \pi_n(S^2)$ .
100. Let  $S^\infty \rightarrow \mathbf{CP}^\infty$  be the Hopf fibration. Using the fact  $S^\infty \sim *$ , prove that  $\pi_n(\mathbf{CP}^\infty) = 0$  for  $n \neq 2$ , and  $\pi_2(\mathbf{CP}^\infty) = \mathbf{Z}$ .
101. Prove that  $\pi_n(\Omega(X)) \cong \pi_{n+1}(X)$  for any  $X$  and  $n \geq 0$ .
102. Prove that if the groups  $\pi_*(B), \pi_*(F)$  are finite (finitely generated), then the groups  $\pi_*(E)$  are finite (finitely generated) as well.

103. Assume that a fiber bundle  $p : E \rightarrow B$  has a *section*, i.e. a map  $s : B \rightarrow E$ , such that  $p \circ s = Id_B$ . Prove the isomorphism  $\pi_n(E) \cong \pi_n(B) \oplus \pi_n(F)$ .

104. Give a construction of a space  $Y$  that  $\pi_n(X, A; x_0) \cong \pi_{n-1}(Y, y_0)$ .

104. State the Freudenthal Theorem. Give a detailed proof that  $\Sigma$  is an isomorphism.

106. Let  $K, L \subset \mathbf{R}^p$  be two finite simplicial complexes of dimensions  $k, l$  respectively. Let  $k + l + 1 < p$ . Prove that the simplicial complexes  $K$  and  $L$  are not linked.

107. Prove that  $\pi_n(S^n) \cong \mathbf{Z}$  for each  $n \geq 1$ .

108. Prove that  $\pi_3(S^2) \cong \mathbf{Z}$ , and the Hopf map  $S^3 \rightarrow S^2$  is a representative of the generator of  $\pi_3(S^2)$ .

109. Define Whitehead product. State basic properties. Prove that if  $\alpha \in \pi_n(X)$ ,  $\beta \in \pi_k(X)$  then  $[\alpha, \beta] = (-1)^{nk}[\beta, \alpha]$ .

110. Define the element  $w \in \pi_{n+k-1}(S^n \vee S^k)$ . Prove that the element  $w \in \pi_{n+k-1}(S^n \vee S^k)$  has infinite order.

111. Prove that the element  $w \in \pi_{n+k-1}(S^n \vee S^k)$  is in a kernel of each of the following homomorphisms:

- (1)  $i_* : \pi_{n+k-1}(S^n \vee S^k) \rightarrow \pi_{n+k-1}(S^n \times S^k)$ ,
- (2)  $pr_*^{(n)} : \pi_{n+k-1}(S^n \vee S^k) \rightarrow \pi_{n+k-1}(S^n)$ ,
- (3)  $pr_*^{(k)} : \pi_{n+k-1}(S^n \vee S^k) \rightarrow \pi_{n+k-1}(S^k)$ .

112. Prove that the element  $w \in \pi_{n+k-1}(S^n \vee S^k)$  is in the kernel of the suspension homomorphism

$$\Sigma : \pi_{n+k-1}(S^n \times S^k) \rightarrow \pi_{n+k}(\Sigma(S^n \times S^k)).$$

113. Prove the isomorphism

$$\pi_{n+k}(S^{n+1} \vee S^{k+1}) \cong \pi_{n+k}(S^{n+1}) \oplus \pi_{n+k}(S^{k+1})$$

114. Let  $\alpha \in \pi_n(X)$ ,  $\beta \in \pi_k(X)$ . Prove that  $[\alpha, \beta] \in \text{Ker } \Sigma$ , where

$$\Sigma : \pi_{n+k-1}(X) \rightarrow \pi_{n+k}(\Sigma X)$$

is the suspension homomorphism.

115. Let  $\iota_{2q} \in \pi_{2q}(S^{2q})$  be a generator represented by the identity map  $S^{2q} \rightarrow S^{2q}$ . Prove that the Whitehead product  $[\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q})$  is a nontrivial element of infinite order.

116. Prove that the suspension  $\Sigma(S^n \times S^k)$  is homotopy equivalent to the wedge  $S^{n+1} \vee S^{k+1} \vee S^{n+k+1}$ .

117. Outline a proof of the following statement:

Let  $X$  be a connected space (not necessarily a  $CW$ -complex) with a base point  $x_0 \in X$ ,  $f : S^n \rightarrow X$  be a map such that  $f(s_0) = x_0$ , where  $s_0$  is a base point of  $S^n$ . Let  $Y = X \cup_f D^{n+1}$ , and  $i : X \rightarrow Y$  be the inclusion. Then the induced homomorphism  $i_* : \pi_q(X, x_0) \rightarrow \pi_q(Y, x_0)$

- (1) is an isomorphism if  $q < n$ ,
- (2) is an epimorphism if  $q = n$ , and
- (3) the kernel  $\text{Ker } i_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, x_0)$  is generated by  $\gamma^{-1}[f]\gamma \in \pi_n(X, x_0)$  where  $\gamma \in \pi_1(X, x_0)$ .

118. Let  $X$  be an  $n$ -connected  $CW$ -complex, and  $Y$  be a  $k$ -connected  $CW$ -complex. Prove that

- $\pi_q(X \vee Y) \cong \pi_q(X) \oplus \pi_q(Y)$  if  $q \leq n + k$ ;
- the group  $\pi_q(X \vee Y)$  contains a subgroup  $\pi_q(X) \oplus \pi_q(Y)$  as a direct summand.

119. Let  $X$  be an  $n$ -connected  $CW$ -complex, and  $Y$  be a  $k$ -connected  $CW$ -complex. Prove that

$$\pi_{n+k+1}(X \vee Y) \cong \pi_{n+k+1}(X) \oplus \pi_{n+k+1}(Y) \oplus [\pi_n(X), \pi_k(Y)].$$

120. Let  $X$  be an  $(n-1)$ -connected  $CW$ -complex. Describe the homotopy group  $\pi_n(X)$ .

121. Compute the homotopy group  $\pi_3(S^2 \vee S^2)$ .

122. Define when a map  $f : X \rightarrow Y$  is a weak homotopy equivalence. Outline the proof that the following two statements are equivalent

- (1) The map  $f : X \rightarrow Y$  is weak homotopy equivalence.
- (2) The induced homomorphism  $f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$  is isomorphism for all  $n$  and  $x_0 \in X$ .

123. Let  $X, Y$  be  $CW$ -complexes. Prove that if a map  $f_* : X \rightarrow Y$  induces isomorphism

$$f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$$

for all  $n \geq 0$  and  $x_0 \in X$ , then  $f$  is a homotopy equivalence.

124. Let  $X$  be a Hausdorff topological space. Prove that there exists a  $CW$ -complex  $K$  and a weak homotopy equivalence  $f : K \rightarrow X$ . Show that the  $CW$ -complex  $K$  is unique up to homotopy equivalence.

125. Let  $X, Y$  be two weak homotopy equivalent spaces. Prove that there exist a  $CW$ -complex  $K$  and maps  $f : K \rightarrow X, g : K \rightarrow Y$  which weak homotopy equivalences.

126. Define an Eilenberg-McLane space. Prove that it does exist and unique up to weak homotopy equivalence.

127. Construct the space  $K(\pi, 1)$ , where  $\pi$  is a finitely generated abelian group.

128. Let  $X = K(\pi, n)$ . Prove that  $\Omega X = K(\pi, n-1)$ .

129. Let  $X$  be a  $CW$ -complex, and  $n \geq 1$ . Construct a  $CW$ -complex  $X_n$  and a map  $f_n : X \rightarrow X_n$  such that

- (1)  $\pi_q(X_n) = \begin{cases} \pi_q(X) & \text{if } q \leq n \\ 0 & \text{else} \end{cases}$
- (2)  $(f_n)_* : \pi_q(X) \rightarrow \pi_q(X_n)$  is isomorphism if  $q \leq n$ .

130. Let  $X$  be a  $CW$ -complex, and  $n \geq 1$ . Construct a  $CW$ -complex  $X|_n$  and a map  $g_n : X|_n \rightarrow X$  such that

- (1)  $\pi_q(X|_n) = \begin{cases} \pi_q(X) & \text{if } q \geq n \\ 0 & \text{else} \end{cases}$
- (2)  $(g_n)_* : \pi_q(X|_n) \rightarrow \pi_q(X)$  is isomorphism if  $q \geq n$ .

131. Let  $X = S^2$ . Prove that  $X|_3 = S^3$ .

132. Let  $X = \mathbf{CP}^n$ . Prove that  $X|_3 = X|_{2n+1} = S^{2n+1}$ .

133. Define the complex  $\mathcal{C}(X)$  and the homology groups  $H_q(X)$ . Calculate the homology groups for  $X = \{pt\}$ .

134. Define chain maps and chain homotopy. Prove that two chain homotopic maps  $\phi, \psi : \mathcal{C} \rightarrow \mathcal{C}'$  induce the same homomorphism in homology groups.

135. Let  $g, h : X \rightarrow Y$  be homotopic maps. Prove that  $g_* = h_* : H_q(X) \rightarrow H_q(Y)$ .

136. Let  $X$  and  $Y$  be homotopy equivalent spaces. Prove that then  $H_q(X) \cong H_q(Y)$  for all  $q$ .

137. Prove that  $H_0(X) \cong \mathbf{Z}$  if  $X$  is a path-connected space.

138. Prove that if  $f : X \rightarrow Y$  is a map of path-connected spaces, then  $f_* : H_0(X) \rightarrow H_0(Y)$  is an isomorphism.

139. Define relative homology groups. State and prove the LES-Lemma.

140. Let  $B \subset A \subset X$  be a triple of spaces. Prove that there is a long exact sequence in homology:

$$\cdots \rightarrow H_q(A, B) \xrightarrow{i_*} H_q(X, B) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A, B) \xrightarrow{i_*} \cdots$$

141. Let  $(X, A)$  be a pair of spaces. Prove that the inclusion  $i : (X, A) \rightarrow (X \cup C(A), C(A))$  induces the isomorphism  $H_q(X, A) \cong H_q(X \cup C(A), C(A)) = H_q(X \cup C(A), v)$ .

142. Define the operation  $\beta : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  (induced by the barycentric subdivision). Prove that the chain map  $\beta : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$  induces the identity homomorphism in homology:

$$Id = \beta_* : H_q(\mathcal{C}(X)) \rightarrow H_q(\mathcal{C}(X)) \quad \text{for each } q \geq 0.$$

143. Define the chain complex  $\mathcal{C}^{\mathbf{U}}(X)$  for a covering  $\mathbf{U}$ . Prove that the inclusion  $\mathcal{C}^{\mathbf{U}}(X) \subset \mathcal{C}(X)$  induces an isomorphism in homology groups.

144. State and prove the Excision Theorem.

145. Let  $X = X_1 \cup X_2$ . Prove that the following sequence of complexes is exact

$$0 \rightarrow \mathcal{C}(X_1 \cap X_2) \xrightarrow{\alpha} \mathcal{C}(X_1) \oplus \mathcal{C}(X_2) \xrightarrow{\beta} \mathcal{C}(X_1) + \mathcal{C}(X_2) \rightarrow 0.$$

146. Let  $X_1, X_2 \subset X$ , and  $X_1 \cup X_2 = X$ ,  $\overset{\circ}{X}_1 \cup \overset{\circ}{X}_2 = X$ . Prove that the chain map

$$\mathcal{C}(X_1) + \mathcal{C}(X_2) \rightarrow \mathcal{C}(X_1 \cup X_2)$$

induces isomorphism in the homology groups.

147. State and prove the Mayer-Vietoris Theorem.

148. Compute homology groups  $H_q(S^n)$ .

149. Let  $X$  be a space. Prove that  $\tilde{H}_{q+1}(\Sigma X) \cong \tilde{H}_q(X)$  for each  $q$ .

150. Let  $A$  be a set of indices, and  $S_\alpha^n$  be a copy of the  $n$ -th sphere,  $\alpha \in A$ . Compute the homology groups  $\tilde{H}_q\left(\bigvee_{\alpha \in A} S_\alpha^n\right)$ .

151. Let  $(X_\alpha, x_\alpha)$  be based spaces,  $\alpha \in A$ . Assume that the pair  $(X_\alpha, x_\alpha)$  is Borsuk pair for each  $\alpha \in A$ . Prove that

$$\tilde{H}_q\left(\bigvee_{\alpha \in A} X_\alpha\right) = \bigoplus_{\alpha \in A} \tilde{H}_q(X_\alpha).$$

152. Let  $f : S^n \rightarrow S^n$  be a map of degree  $d = \deg f$ . Prove that  $f_* : H_n(S^n) \rightarrow H_n(S^n)$  is a multiplication by  $d$ .

153. Let  $g : \bigvee_{\alpha \in A} S_\alpha^n \xrightarrow{g} \bigvee_{\beta \in B} S_\beta^n$  be a map. Prove that the homomorphism

$$\bigoplus_{\alpha \in A} \mathbf{Z}(\alpha) = H_n\left(\bigvee_{\alpha \in A} S_\alpha^n\right) \xrightarrow{g_*} H_n\left(\bigvee_{\beta \in B} S_\beta^n\right) = \bigoplus_{\beta \in B} \mathbf{Z}(\beta)$$

is given by multiplication with matrix  $\{d_{\alpha\beta}\}_{\alpha \in A, \beta \in B}$ , where  $d_{\alpha\beta} = \deg g_{\alpha\beta}$ . (Define the maps  $g_{\alpha\beta}$ .)

154. Define the cellular chain complex  $\mathcal{E}(X)$ . Prove that the following composition is trivial

$$\mathcal{E}_{q+1}(X) \xrightarrow{\partial_{q+1}} \mathcal{E}_q(X) \xrightarrow{\partial_q} \mathcal{E}_{q-1}(X).$$

155. Prove that there is an isomorphism  $H_q(\mathcal{E}(X)) \cong H_q(X)$  for each  $q$  and any  $CW$ -complex  $X$ .

156. Let  $X$  be a  $CW$ -complex, and  $e^q$  be a  $q$ -cell and  $\sigma^{q-1}$  be a  $(q-1)$ -cell of  $X$ . Define the incidence coefficient  $[e^q : \sigma^{q-1}]$ . Prove that the boundary operator  $\partial_q : \mathcal{E}_q(X) \rightarrow \mathcal{E}_{q-1}(X)$  is given by the formula:

$$\partial_q(e^q) = \sum_{j \in E_{q-1}} [e^q : \sigma_j^{q-1}] \sigma_j^{q-1}.$$

157. Let  $A : S^n \rightarrow S^n$  be the antipodal map,  $A : x \mapsto -x$ , and  $\iota_n \in \pi_n(S^n)$  be the generator represented by the identity map  $S^n \rightarrow S^n$ . Prove that the homotopy class  $[A] \in \pi_n(S^n)$  is equal to

$$[A] = \begin{cases} \iota_n, & \text{if } n \text{ is odd,} \\ -\iota_n, & \text{if } n \text{ is even.} \end{cases}$$

158. Let  $e^0, \dots, e^n$  be the cells in the standard cell decomposition of  $\mathbf{RP}^n$ . Prove that

$$[e^q : e^{q-1}] = \begin{cases} 2 & \text{if } q \text{ is odd,} \\ 0, & \text{if } q \text{ is even.} \end{cases}$$

159. Compute the homology groups  $H_q(\mathbf{RP}^n)$ ,  $H_q(\mathbf{CP}^n)$ .

160. Compute the homology groups  $H_q((\mathbf{RP}^n)^{\#k})$  and  $H_q((\mathbf{CP}^n)^{\#k})$

161. Compute the homology groups  $H_q(\mathbf{RP}^{2n} \# \mathbf{CP}^n)$ .

162. Prove that there is no map  $f : D^n \rightarrow S^{n-1}$  so that the restriction  $f|_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$  has nonzero degree.

163. Let  $X$  be a topological space,  $\alpha \in H_q(X)$ . Prove that there exist a  $CW$ -complex  $K$ , a map  $f : K \rightarrow X$ , an element  $\beta \in H_q(K)$  such that  $f_*(\beta) = \alpha$ .

164. Let  $f : X \rightarrow Y$  be a weak homotopy equivalence. Prove that the induced homomorphism  $f_* : H_q(X) \rightarrow H_q(Y)$  is an isomorphism for all  $q \geq 0$ .

165. Show that the spaces  $\mathbf{CP}^\infty \times S^3$  and  $S^2$  have isomorphic homotopy groups and that they are not homotopy equivalent.

166. Show that the spaces  $\mathbf{RP}^n \times S^m$  and  $S^n \times \mathbf{RP}^m$  ( $n \neq m$ ) have isomorphic homotopy groups and they are not homotopy equivalent.

167. Show that the spaces  $S^1 \vee S^1 \vee S^2$  and  $S^1 \times S^1$  have the same homology groups and different homotopy groups.

168. Show that the projection

$$S^1 \times S^1 \xrightarrow{\text{projection}} (S^1 \times S^1)/(S^1 \vee S^1) = S^2$$

induces trivial homomorphism in homotopy groups.

169. Define the Hurewicz homomorphism  $h : \pi_n(X, x_0) \rightarrow H_n(X)$ . Prove that  $h$  is a homomorphism.

170. Let  $x_0, x_1 \in X$ , and  $\gamma : I \rightarrow X$  be a path connecting the points  $x_0, x_1$ :  $\gamma(0) = x_0$ , and  $\gamma(1) = x_1$ . The path  $\gamma$  determines the isomorphism  $\gamma_{\#} : \pi_n(X, x_0) \rightarrow \pi_n(X, x_1)$ . Prove that the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\gamma_{\#}} & \pi_n(X, x_1) \\ & \searrow h & \swarrow h \\ & H_n(X) & \end{array}$$

171. (Hurewicz Theorem) Let  $(X, x_0)$  be a based space, such that

$$\pi_0(X, x_0) = 0, \pi_1(X, x_0) = 0, \dots, \pi_{n-1}(X, x_0) = 0,$$

where  $n \geq 2$ . Prove that

$$H_1(X) = 0, H_2(X) = 0, \dots, H_{n-1}(X) = 0,$$

and the Hurewicz homomorphism  $h : \pi_{n-1}(X, x_0) \rightarrow H_n(X)$  is an isomorphism.

172. Let  $X$  be a simply-connected  $CW$ -complex with  $\tilde{H}_n(X) = 0$  for all  $n$ . Prove that  $X$  is contractible.
173. Let  $X$  be a simply connected space, and  $H_1(X) = 0, H_2(X) = 0 \dots H_{n-1}(X) = 0$ . Prove that  $\pi_1(X) = 0, \pi_2(X) = 0 \dots \pi_{n-1}(X) = 0$  and the Hurewicz homomorphism  $h : \pi_n(X, x_0) \rightarrow H_n(X)$  is an isomorphism.
174. Consider the map

$$g : S^{2n-2} \times S^3 \xrightarrow{\text{proj}} (S^{2n-2} \times S^3) / (S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbf{CP}^n.$$

Prove that  $g$  induces trivial homomorphism in homology and homotopy groups, however  $g$  is not homotopic to a constant map.

175. Let  $X$  be a connected space. Prove that the Hurewicz homomorphism  $h : \pi_1(X, x_0) \rightarrow H_1(X)$  is epimorphism, and the kernel of  $h$  is the commutator  $[\pi_1(X, x_0), \pi_1(X, x_0)] \subset \pi_1(X, x_0)$ .
176. State the relative version of the Hurewicz Theorem. State and prove the Whitehead Theorem-II. Let  $X, Y$  be simply connected spaces and  $f : X \rightarrow Y$  be a map which induces isomorphism  $f_* : H_q(X) \rightarrow H_q(Y)$  for all  $q \geq 0$ . Prove that  $f$  is weak homotopy equivalence.
177. Define homology and cohomology groups with coefficients in an abelian group  $G$ . Compute the groups  $H_q(\mathbf{RP}^n; \mathbf{Z}/p)$ ,  $H^q(\mathbf{RP}^n; \mathbf{Z}/p)$  for any prime  $p$ .
178. Consider the short exact sequence  $0 \rightarrow \mathbf{Z} \xrightarrow{m} \mathbf{Z} \rightarrow \mathbf{Z}/2 \rightarrow 0$ . Compute the connecting homomorphisms

$$\partial = \beta^m : H^q(\mathbf{RP}^n; \mathbf{Z}/2) \rightarrow H^{q+1}(\mathbf{RP}^n; \mathbf{Z})$$

- 179.\* Let  $G$  be an abelian group,  $0 \rightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \rightarrow 0$ , be a free resolution of  $G$ , and  $H$  be an arbitrary abelian group. Prove that the sequence

$$0 \rightarrow \text{Ker}(\beta \otimes 1) \rightarrow R \otimes H \xrightarrow{\beta \otimes 1} F \otimes H \xrightarrow{\alpha \otimes 1} G \otimes H \rightarrow 0$$

is exact.

- 180.\* Prove that the group  $\text{Tor}(G, H)$  is well-defined, i.e. it does not depend on the choice of resolution.
- 181.\* Let  $G, H$  be abelian groups. Prove that there is a canonical isomorphism  $\text{Tor}(G, H) \cong \text{Tor}(H, G)$ .

182. Let  $F$  be a free abelian group. Show that  $\text{Tor}(F, G) = 0$  for any abelian group  $G$ .

183. Let  $G$  be an abelian group. Denote  $T(G)$  a maximal torsion subgroup of  $G$ . Show that  $\text{Tor}(G, H) \cong T(G) \otimes T(H)$  for finite generated abelian groups  $G, H$ . Give an example of abelian groups  $G, H$ , so that  $\text{Tor}(G, H) \neq T(G) \otimes T(H)$ .

184. Let  $X$  be a space,  $G$  be an abelian group. Prove that there is a split short exact sequence

$$0 \rightarrow H_q(X) \otimes G \rightarrow H_q(X; G) \rightarrow \text{Tor}(H_{q-1}(X), G) \rightarrow 0$$

185\*. Let  $G$  be an abelian group,  $0 \rightarrow R \xrightarrow{\beta} F \xrightarrow{\alpha} G \rightarrow 0$  be a free resolution, and let  $H$  be an abelian group. Prove that the following sequence is exact:

$$0 \leftarrow \text{Coker } \beta^\# \leftarrow \text{Hom}(R, H) \xleftarrow{\beta^\#} \text{Hom}(F, H) \xleftarrow{\alpha^\#} \text{Hom}(G, H) \leftarrow 0.$$

186\*. Prove that the group  $\text{Ext}(G, H)$  is well defined, i.e. it does not depend on the choice of free resolution of  $G$ .

187\*. Let  $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$  be a short exact sequence of abelian groups. Prove that it induces the following exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Hom}(G'', H) \rightarrow \text{Hom}(G, H) \rightarrow \text{Hom}(G', H) \rightarrow \\ \text{Ext}(G'', H) \rightarrow \text{Ext}(G, H) \rightarrow \text{Ext}(G', H) \rightarrow 0 \end{aligned}$$

188. Prove that  $\text{Ext}(\mathbf{Z}, H) = 0$  for any group  $H$ .

189. Prove the isomorphisms:  $\text{Ext}(\mathbf{Z}/m, \mathbf{Z}/n) \cong \mathbf{Z}/m \otimes \mathbf{Z}/n$ ,  $\text{Ext}(\mathbf{Z}/m, \mathbf{Z}) \cong \mathbf{Z}/m$ .

190. Let  $X$  be a space,  $G$  an abelian group. Prove that there is a split exact sequence

$$0 \rightarrow \text{Ext}(H_{q-1}(X), G) \rightarrow H^q(X; G) \rightarrow \text{Hom}(H_q(X), G) \rightarrow 0$$

for any  $q \geq 0$ .

191. Let  $X$  be a space, and  $G$  an abelian group. Prove that there is a split exact sequence

$$0 \rightarrow H^q(X; \mathbf{Z}) \otimes G \rightarrow H^q(X; G) \rightarrow \text{Tor}(H^{q+1}(X; \mathbf{Z}), G) \rightarrow 0$$

for any  $q \geq 0$ .

192. Let  $G$  be a finitely generated abelian group. Let  $F(G)$  be the maximum free abelian subgroup of  $G$ , and  $T(G)$  be the maximum torsion subgroup. Let  $X$  be a space such that the groups  $H_q(X)$  are finitely generated for all  $q$ . Prove that  $H^q(X; \mathbf{Z})$  are also finitely generated and  $H^q(X; \mathbf{Z}) \cong F(H_q(X; \mathbf{Z})) \oplus T(H_{q-1}(X; \mathbf{Z}))$ .

193. Let  $F$  be  $\mathbf{Q}$ ,  $\mathbf{R}$  or  $\mathbf{C}$ . Prove that

$$H_q(X; F) = H_q(X) \otimes F, \quad H^q(X; F) = \text{Hom}(H_q(X), F).$$

194. Let  $X$  be a finite  $CW$ -complex, and  $\mathbf{F}$  be a field. Prove that the number

$$\chi(X)_{\mathbf{F}} = \sum_{q \geq 0} (-1)^q \dim H_q(X; \mathbf{F})$$

does not depend on the field  $\mathbf{F}$  and is equal to the Euler characteristic

$$\chi(X) = \sum_{q \geq 0} (-1)^q \{ \# \text{ of } q\text{-cells of } X \}.$$

195. Let a finite  $CW$ -complex  $X$  be a union of two  $CW$ -subcomplexes:  $X = X_1 \cup X_2$ , where  $X_1 \cap X_2 \subset X$  is a  $CW$ -subcomplex as well. Prove that

$$\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

196. Let  $\mathcal{C}_*$  and  $\mathcal{C}'_*$  be two chain complexes. Define the tensor product  $\bar{\mathcal{C}}_* = \mathcal{C}_* \otimes \mathcal{C}'_*$ . Prove that  $\bar{\partial}_{q+1}\bar{\partial}_q = 0$ .

197. Let  $\mathcal{E}_* = \mathcal{E}_*(X)$ ,  $\mathcal{E}'_* = \mathcal{E}'_*(X')$ . Define the complexes  $\mathcal{E}_*(r)$ ,  $\mathcal{E}'_*(s)$  and compute the homology groups of the tensor product of these chain complexes:  $\mathcal{E}_*(r) \otimes \mathcal{E}'_*(s)$ .

198. Use the result of # 197 to prove the Künneth formula for homology groups:

$$0 \rightarrow \bigoplus_{r+s=q} H_r(X) \otimes H_s(X') \rightarrow H_q(X \times X') \rightarrow \bigoplus_{r+s=q-1} \text{Tor}(H_r(X), H_s(X')) \rightarrow 0$$

199. Outline the proof of the Künneth formula for cohomology groups:

$$0 \rightarrow \bigoplus_{r+s=q} H^r(X) \otimes H^s(X') \rightarrow H^q(X \times X') \rightarrow \bigoplus_{r+s=q+1} \text{Tor}(H^r(X), H^s(X')) \rightarrow 0.$$

200. Let  $F$  be a field. Prove that

$$H_q(X \times X'; F) \cong \bigoplus_{r+s=q} H_r(X; F) \otimes H_s(X'; F),$$

$$H^q(X \times X'; F) \cong \bigoplus_{r+s=q} H^r(X; F) \otimes H^s(X'; F).$$

201. Let  $\beta_q(X) = \text{Rank} H_q(X)$  be the Betti number of  $X$ . Prove that

$$\beta_q(X \times X') = \sum_{r+s=q} \beta_r(X) \beta_s(X').$$

202. Let  $X$ ,  $X'$  be such spaces that their Euler characteristics  $\chi(X)$ ,  $\chi(X')$  are finite. Prove that  $\chi(X \times X') = \chi(X) \cdot \chi(X')$ .

203. Prove the Lefschetz Theorem: Let  $X$  be a finite  $CW$ -complex,  $f : X \rightarrow X$  be a map such that  $\text{Lef}(f) = 0$ . Then  $f$  has a fixed point, i.e. such point  $x_0 \in X$  that  $f(x_0) = x_0$ .

204. Let  $X$  be a finite contractible  $CW$ -complex. Prove that any map  $f : X \rightarrow X$  has a fixed point.

205. Define a flow of homeomorphisms  $\phi_t : X \rightarrow X$ . Let  $X$  be a finite  $CW$ -complex with  $\chi(X) \neq 0$ , and  $\phi_t : X \rightarrow X$  be a flow. Prove that there exists a point  $x_0 \in X$  so that  $\phi_t(x_0) = x_0$  for all  $t \in \mathbf{R}$ .

206. Let  $f : \mathbf{RP}^{2n} \rightarrow \mathbf{RP}^{2n}$  be a map. Prove that  $f$  always has a fixed point. Give an example that the above statement fails for a map  $f : \mathbf{RP}^{2n+1} \rightarrow \mathbf{RP}^{2n+1}$ .

207. Let  $n \neq k$ . Prove that  $\mathbf{R}^n$  is not homeomorphic to  $\mathbf{R}^k$ .

208. Let  $f : S^n \rightarrow S^n$  be a map, and  $\deg(f)$  be the degree of  $f$ . Prove that  $\text{Lef}(f) = 1 + (-1)^n \deg(f)$ .

209. Prove that there is no tangent vector field  $v(x)$  on the sphere  $S^{2n}$  such that  $v(x) \neq 0$  for all  $x \in S^{2n}$ . Construct everywhere non-zero vector field  $v$  on  $S^{2n+1}$ .

210. Let  $K \subset S^n$  be homeomorphic to the cube  $I^k$ ,  $0 \leq k \leq n$ . Prove that  $\tilde{H}_q(S^n \setminus K) = 0$  for all  $q \geq 0$ .

211. Let  $S^k \subset S^n$ ,  $0 \leq k \leq n-1$ . Prove that

$$\tilde{H}_q(S^n \setminus S^k) \cong \begin{cases} \mathbf{Z}, & \text{if } q = n - k - 1, \\ 0 & \text{if } q \neq n - k - 1. \end{cases}$$

212. State and prove the Jordan-Brouwer Theorem.

213. State and prove the Brouwer Invariance Domain Theorem.

214. Let  $(X, A)$  be a  $CW$ -pair. Prove that the group  $H^1(X, A; \mathbf{Z})$  is a free abelian group.

215. Define the cup-product in cohomology. Prove that  $\delta(\phi \cup \psi) = (\delta\phi) \cup \psi + (-1)^k \phi \cup (\delta\psi)$  where  $\phi \in C^k(X)$ ,  $\psi \in C^l(X)$ .

216. Compute the cup product of  $H^*(\mathbf{RP}^2; \mathbf{Z}/2)$ ,  $H^*(M_g^2; \mathbf{Z})$ .

217. Prove that  $\alpha\beta = (-1)^{kl}\beta\alpha$  if  $\alpha \in H^k(X)$ ,  $\beta \in H^l(X)$ .

218. Define the external product

$$\mu : H^*(X; R) \otimes H^*(Y; R) \rightarrow H^*(X \times Y; R).$$

Define the ring structure on  $H^*(X; R) \otimes H^*(Y; R)$ . Prove that the external product  $\mu : H^*(X; R) \otimes H^*(Y; R) \rightarrow H^*(X \times Y; R)$  induces a ring isomorphism provided that  $H^q(Y; R)$  are free  $R$ -modules for all  $q$ .

219. Let  $\Delta : X \rightarrow X \times X$  be a diagonal map. Prove that the homomorphism

$$H^k(X; R) \otimes H^l(X; R) \xrightarrow{\mu} H^{k+l}(X \times X; R) \xrightarrow{\Delta^*} H^{k+l}(X; R)$$

coincides with the cup-product, i.e. that  $\Delta^*(\mu(\alpha \otimes \beta)) = \alpha \cup \beta$ .

220. Prove that  $H^*(\mathbf{RP}^n; \mathbf{Z}/2) \cong \mathbf{Z}/2[x]/x^{n+1}$ .

221. Prove that  $H^*(\mathbf{CP}^n; \mathbf{Z}) \cong \mathbf{Z}[y]/y^{n+1}$ .

222. Prove that any map  $f : \mathbf{CP}^{2k} \rightarrow \mathbf{CP}^{2k}$  has a fixed point.

223. Prove that if  $\mathbf{R}^n$  is a real division algebra, then  $n$  is a power of two.

224. State the Poincaré Duality Theorem. Compute the Poincaré Duality for  $M_g^2$ .

225. Prove that the odd-dimensional manifold has zero Euler characteristic.

226. Prove that  $\langle \alpha \cup \beta, \mu \rangle = \langle \beta, \mu \cap \alpha \rangle$ .

227. Let  $M^{4k}$  be a compact oriented manifold, and  $V = H^{2k}(M^{4k}; \mathbf{Z})/\text{Tor}$ . Use the Poincaré duality to prove that the pairing

$$\mu(\alpha, \beta) = \langle \alpha \cup \beta, [M^{4k}] \rangle$$

defines a nondegenerated quadratic form on  $V$ . Compute the index of this quadratic form for  $\mathbf{CP}^{2n}$ .

228. Use Poincaré duality to prove that  $H^*(\mathbf{CP}^n; \mathbf{Z}) \cong \mathbf{Z}[x]/x^{n+1}$ .

229. Use Poincaré duality to prove that  $H^*(\mathbf{RP}^n; \mathbf{Z}/2) \cong \mathbf{Z}/2[x]/x^{n+1}$ .

230. Let  $f : \mathbf{CP}^{2n} \rightarrow \mathbf{CP}^{2n}$  be a map. Show that  $f$  has a fixed point.

231. Compute the ring structure  $H^*(\mathbf{RP}^n; \mathbf{Z}/2^k)$ .

232. Let  $n > k$ . Prove that there is no map  $f : \mathbf{RP}^n \rightarrow \mathbf{RP}^k$  which induces a nontrivial ring homomorphism  $f^* : H^*(\mathbf{RP}^k; \mathbf{Z}/2) \rightarrow H^*(\mathbf{RP}^n; \mathbf{Z}/2)$ .

**233.** Let a map  $h : \mathbf{RP}^{n-1} \times \mathbf{RP}^{n-1} \rightarrow \mathbf{RP}^{n-1}$ , be such that the induced homomorphism

$$h^* : H^*(\mathbf{RP}^{n-1}; \mathbf{Z}/2) \rightarrow H^*(\mathbf{RP}^{n-1} \times \mathbf{RP}^{n-1}; \mathbf{Z}/2)$$

takes generator  $y \in H^1(\mathbf{RP}^{n-1}; \mathbf{Z}/2)$  to the sum of generators:  $h^*(y) = x_1 \otimes 1 + 1 \otimes x_2$ . Prove that  $n$  must be a power of 2.

**234.** Prove that  $\mathbf{RP}^3$  is not homotopy equivalent to  $S^3 \vee \mathbf{RP}^2$ .

**235.** Define the Hopf invariant  $h(\lambda)$  of an element  $\lambda \in \pi_{4q-1}(S^{2q})$ .

**236.** Prove that  $h(\lambda_1 + \lambda_2) = h(\lambda_1) + h(\lambda_2)$ .

**237.** Prove that there is an element in  $\pi_{4n-1}(S^{2n})$  with the Hopf invariant 2. State and prove the theorem that the group  $\pi_{4n-1}(S^{2n})$  is infinite.

**238.** Prove that  $h([\iota_{2q}, \iota_{2q}]) = 2$ , where  $\iota_{2q} \in \pi_{2q}(S^{2q})$  is the standard generator.

**239.** Define a *cohomology operation*. Give examples.

**240.** Define a canonical *fundamental class*

$$\iota_n \in \text{Hom}(H_n(K(\pi, n); \mathbf{Z}), \pi).$$

**241.** Let  $\pi, \pi'$  be abelian groups. Prove that there is a bijection

$$[K(\pi, n), K(\pi', n)] \leftrightarrow \text{Hom}(\pi, \pi').$$

**242.** Let  $\pi$  be an abelian group and  $n$  be a positive integer. Prove that the homotopy type of the Eilenberg-McLane space  $K(\pi, n)$  is completely determined by the group  $\pi$  and the integer  $n$ .

**243.** Prove that there is a bijection

$$\mathcal{O}(\pi, n; \pi', n') \leftrightarrow H^{n'}(K(\pi, n), \pi')$$

given by the formula  $\theta \leftrightarrow \theta(\iota_n)$ .

**244.** Let  $Y$  be a homotopy simple space,  $(B, A)$  a *CW*-pair and  $X^n = B^{(n)} \cup A$  for  $n = 0, 1, \dots$ . Define the obstruction cochain

$$c(f) \in \mathcal{E}^{n+1}(B, A; \pi_n(Y)) = \text{Hom}(\mathcal{E}_{n+1}(B, A), \pi_n(Y)).$$

Prove that  $c(f)$  is a cocycle.

**245.** Let  $Y$  be a homotopy simple space,  $(B, A)$  a *CW*-pair and  $X^n = B^{(n)} \cup A$  for  $n = 0, 1, \dots$ . Prove that a map  $f : X^n \rightarrow Y$  can be extended to a map  $\tilde{f} : X^{n+1} \rightarrow Y$  if and only if  $c(f) = 0$ .

**246.** Define  $d(f, g) \in \mathcal{E}^n(B, A; \pi_n(Y))$ . Prove the formula:  $\delta d(f, g) = c(g) - c(f)$ .

**247.** Let  $Y$  be a homotopy simple space,  $(B, A)$  a *CW*-pair and  $X^n = B^{(n)} \cup A$  for  $n = 0, 1, \dots$ . Let  $f : X^n \rightarrow Y$  be a map, and  $d \in \mathcal{E}^n(B, A; \pi_n(Y))$  is a cochain. Prove that there exists a map  $g : X^n \rightarrow Y$  such that  $f|_{X^{n-1}} = g|_{X^{n-1}}$  and  $d(f, g) = d$ .

**248.** Let  $Y$  be a homotopy simple space,  $(B, A)$  a *CW*-pair and  $X^n = B^{(n)} \cup A$  for  $n = 0, 1, \dots$ . Assume  $f : X^n \rightarrow Y$  is a map. Prove that there exists a map  $g : X^{n+1} \rightarrow Y$  such that  $g|_{X^{n-1}} = f|_{X^{n-1}}$  if and only if  $[c(f)] = 0$  in  $H^{n+1}(B, A; \pi_n Y)$ .

**249.** Prove the following result

**Theorem.** Let  $f, g : K \rightarrow Y$  be two maps, where  $K$  is a *CW*-complex and  $Y$  is homotopy-simple space. Assume that  $f|_{K^{(n-1)}} = g|_{K^{(n-1)}}$ . Then the cohomology class  $[d(f, g)] \in H^n(K, \pi_n Y)$  vanishes if and only if there exists a homotopy between the maps  $f|_{K^{(n)}}$  and  $g|_{K^{(n)}}$  relative to the skeleton  $K^{(n-2)}$ .

**250.** Prove the following result:

**Theorem.** There is a bijection

$$[X, K(\pi, n)] \leftrightarrow H^n(X; \pi).$$

given by the formula  $[f] \mapsto f^* \iota_n$ .

**251.** Consider a  $k$ -torus  $T^k$ . We identify  $T^k$  with the quotient space  $\mathbf{R}^k / \sim$ , where two vectors  $\vec{x} \sim \vec{y}$  if and only if all coordinates of the vector  $\vec{x} - \vec{y}$  are integers. It is easy to see that a linear map  $\tilde{f} : \mathbf{R}^k \rightarrow \mathbf{R}^\ell$  given by an  $k \times \ell$ -matrix  $A$  with integral entries descends to a map  $f : T^k \rightarrow T^\ell$ . Prove that any map  $g : T^k \rightarrow T^\ell$  is homotopic to a linear map as above.

**252.** Prove the following result:

**Theorem.** Let  $X$  be an  $n$ -dimensional  $CW$ -complex. Then there is a bijection:

$$H^n(X; \mathbf{Z}) \cong [X, S^n].$$

**253.** Prove that any  $K(\mathbf{Z}_2, n)$  is infinite dimensional space for each  $n \geq 1$ .

**254.** Let  $M$  be a simply-connected compact closed manifold with  $\dim M = 3$ . Prove that  $M$  is homotopy equivalent to  $S^3$ .

**255.** Let  $h : S^3 \rightarrow S^2$  be the Hopf map. Let  $\lambda \geq 1$  be an integer. Define a map

$$f_\lambda : S^3 \xrightarrow{\lambda} \underbrace{S^3 \vee \dots \vee S^3}_\lambda \xrightarrow{h \vee \dots \vee h} S^2.$$

Prove that the space  $X_\lambda = S^2 \cup_{f_\lambda} D^4$  is homotopy equivalent to a closed compact manifold of dimension four if and only if  $\lambda = 1$ .

**256.** Let  $D^3 \subset T^3$  and  $c : T^3 \rightarrow S^3$  be a map which collapses a complement of  $D^3 \subset T^3$  to a point. Prove that the map  $g : T^3 \xrightarrow{c} S^3 \xrightarrow{h} S^2$  (where  $h : S^3 \rightarrow S^2$  is the Hopf map) induces trivial homomorphism on homology and homotopy, but is not homotopic to a constant map.

**257.** Assume a  $CW$ -complex  $X$  contains  $S^1$  such that the inclusion  $i : S^1 \subset X$  induces an injection  $i_* : H_1(S^1; \mathbf{Z}) \rightarrow H_1(X; \mathbf{Z})$  with image a direct summand of  $H_1(X; \mathbf{Z})$ . Prove that  $S^1$  is a retract of  $X$ .

**258.** Two questions:

- (a) Show that there is no map from  $\mathbf{CP}^2$  to itself of degree  $-1$ .
- (b) Show that there is no map from  $\mathbf{CP}^2 \times \mathbf{CP}^2$  to itself of degree  $-1$ .