Vanishing and Estimation Results for Betti numbers

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September 30, 2020

Geometry and Topology Seminar,
University of Oregon

This talk is based on joint work with Peter Petersen

The Bochner Technique

Let (M^n, g) be a compact, connected, oriented Riemannian manifold.

A 1-form $\omega \in \Omega^1(M)$ satisfies the Bochner formula

$$\Delta\omega = (dd^* + d^*d)\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega^\#, \cdot)$$

If ω harmonic, $\Delta\omega = 0$, then

$$\Delta\frac{1}{2}|\omega|^2 = |\nabla\omega|^2 + \mathrm{Ric}(\omega^\#, \omega^\#)$$

Bochner, 1948: Suppose ω harmonic.

If $\mathrm{Ric} \geq 0$, then $\nabla\omega = 0$.

If $\mathrm{Ric} > 0$, then $\omega = 0$. Hence, by Hodge theory, $b_1(M) = 0$.

The Bochner Technique: p-forms

Bochner Formula for p-forms

$$\Delta\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega)$$

where

$$\mathrm{Ric}(\omega)(X_1, \dots, X_p) = \sum_{i=1}^{k}\sum_{j=1}^{n}(R(X_i, e_j)\omega)(X_1, \dots, e_j, \dots, X_p)$$

If ω harmonic, then

$$\Delta\frac{1}{2}|\omega|^2 = |\nabla\omega|^2 + g(\mathrm{Ric}(\omega), \omega)$$

The Bochner Technique: p-forms

Bochner Formula for p-forms

$$\Delta\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega)$$

where

$$\mathrm{Ric}(\omega)(X_1, \ldots, X_p) = \sum_{i=1}^{k} \sum_{j=1}^{n} (R(X_i, e_j)\omega)(X_1, \ldots, e_j, \ldots, X_k)$$

Basic observation: $R(X, Y) \in \mathfrak{so}(TM) \cong \Lambda^2 TM$
Curvature operator:

$$\mathfrak{R} \colon \Lambda^2 TM \to \Lambda^2 TM$$
$$g(\mathfrak{R}(X \wedge Y), Z \wedge W) = \mathrm{Rm}(X, Y, Z, W)$$

The Bochner Technique: p-forms

Let $\lambda_1 \leq \ldots \leq \lambda_{\binom{n}{2}}$ denote the eigenvalues of the curvature operator $\mathfrak{R}$ and let $\{\Xi_\alpha\}$ be an orthonormal eigenbasis

$$\mathrm{Ric}(\omega)(X_1, \ldots, X_p) = \sum_{i=1}^{k} \sum_{j=1}^{n} (R(X_i, e_j)\omega)(X_1, \ldots, e_j, \ldots, X_k)$$

Proposition (Poor, 1980)

$$g(\mathrm{Ric}(\omega), \omega) = \sum_{\alpha} \lambda_\alpha |\Xi_\alpha \omega|^2$$

where

$$(\Xi\omega)(X_1, \ldots, X_p) = -\sum_{k=1}^{p} \omega(X_1, \ldots, \Xi X_k, \ldots, X_p).$$

The Bochner Technique: p-forms

Curvature term in Bochner formula $\Delta \frac{1}{2}|\omega|^2 = |\nabla \omega|^2 + g(\mathrm{Ric}(\omega), \omega)$

$$g(\mathrm{Ric}(\omega), \omega) = \sum_{\alpha} \lambda_\alpha |\Xi_\alpha \omega|^2$$

Consequences: Suppose ω is harmonic.

1. D. Meyer, 1971: If $\lambda_\alpha > 0$, then $\omega = 0$, hence $b_p(M) = 0$ for $p \neq 0, n$
2. Gallot-Meyer, 1975: If $\lambda_\alpha \geq 0$, then ω is parallel
3. Gallot, 1981: If $\lambda_\alpha \geq \kappa$, $(\kappa \leq 0)$, $\mathrm{diam}(M) \leq D$, then

$$b_p(M) \leq \binom{n}{p} \exp\left(C\left(n, \kappa D^2\right) \cdot \sqrt{-\kappa D^2 p(n-p)} \right)$$

4. Micallef-Wang, 1993:
   If M^{2n} has with positive isotropic curvature, then $b_2(M) = 0$

Ricci flow results

1. Hamilton (1982, 1986), Chen (1991), Böhm-Wilking (2008)
   If (M, g) has 2-positive curvature operator, $\lambda_1 + \lambda_2 > 0$, then M is diffeomorphic to a space form.
2. Brendle-Schoen (2009)
   If $M \times \mathbb{R}^2$ has positive isotropic curvature, then M is diffeomorphic to a space form.
3. Brendle (2008)
   If $M \times \mathbb{R}$ has positive isotropic curvature, then M is diffeomorphic to a space form.
4. Bamler-Cabezas-Rivas-Wilking (2019)
   For $n \in \mathbb{N}$, $D, v_0 > 0$ there is $\varepsilon(n, D, v_0) > 0$ such that if $\lambda_\alpha \geq -\varepsilon$, $\mathrm{Vol}_g(M) \geq v_0$, $\mathrm{diam}(M) \leq D$, then M has a metric with nonnegative curvature operator

Vanishing and Estimation Results for Betti numbers

Theorem (Petersen-W, 2019)

Let $n \geq 3$ and let (M^n, g) be a compact, connected Riemannian manifold. Let $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$, $\kappa \leq 0$ and $D > 0$. There is $C\left(n, \kappa D^2\right) > 0$ such that if

$$\operatorname{diam} M \leq D \text{ and } \frac{\lambda_1 + \ldots + \lambda_{n-p}}{n-p} \geq \kappa,$$

then

$$b_p(M) \leq \binom{n}{p} \exp\left(C\left(n, \kappa D^2\right) \cdot \sqrt{-\kappa D^2 p(n-p)} \right)$$

There is $\varepsilon(n) > 0$ such that $\kappa D^2 \geq -\varepsilon(n)$ implies $b_p(M) \leq \binom{n}{p}$.

Moreover, suppose $\omega \in \Omega^p$ is harmonic.

1. *If $\lambda_1 + \ldots + \lambda_{n-p} \geq 0$, then ω is parallel*
2. *If $\lambda_1 + \ldots + \lambda_{n-p} > 0$, then $\omega = 0$ and hence $b_p(M) = b_{n-p}(M) = 0$*

Vanishing and Estimation Results for Betti numbers

Corollary (Petersen-W, 2019)

If $\lambda_1 + \ldots + \lambda_{\lceil \frac{n}{2} \rceil} > 0$, *then* M *is a homology sphere*

1. The curvature conditions $\lambda_1 + \ldots + \lambda_{n-p} \geq 0$ are (typically) *not* preserved by the Ricci flow, e.g. $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$ is not preserved (Böhm-Wilking, 2008)
2. Micallef-Moore, 1988:
   If M is simply connected and has positive isotropic curvature, then M is a homotopy sphere
3. $\{(M, g) \mid \lambda_1 + \ldots + \lambda_{n-p} > 0\}$ (typically) overlaps with $\{(M, g) \mid$ positive isotropic curvature$\}$ but neither class is contained in the other
4. Gromov's bound on the Betti numbers (1981)

Idea of the proof

Recall:

$$g(\mathrm{Ric}(\omega), \omega) = \sum_{\alpha} \lambda_\alpha |\Xi_\alpha \omega|^2$$

Key estimates:

$$|\Xi_\alpha \omega|^2 \leq \min\{p, n-p\}|\omega|^2, \quad |\Xi_\alpha|^2 = 1$$

$$\sum_{\alpha} |\Xi_\alpha \omega|^2 = p(n-p)|\omega|^2$$

Idea: Pick normal form for Ξ

Consequence: Let $\kappa \leq 0$.

$$\text{If} \quad \frac{\lambda_1 + \ldots + \lambda_{n-p}}{n-p} \geq \kappa, \quad \text{then} \quad g(\mathrm{Ric}(\omega), \omega) \geq \kappa p(n-p)|\omega|^2.$$

The work of P. Li and Gallot implies the estimation theorem.

Kähler manifolds

Let (M, g) be a compact Kähler manifold of complex dimension m, i.e. $\mathrm{Hol}(g) \subset U(m)$.

Riemannian curvature operator

$$\mathfrak{R}_{|\mathfrak{u}(m)} \colon \mathfrak{u}(m) \to \mathfrak{u}(m)$$
$$\mathfrak{R}_{|\mathfrak{u}(m)^\perp} = 0$$

In particular, $\dim \ker \mathfrak{R} \geq m(m-1)$.

If $\lambda_1 + \ldots + \lambda_{2m-p} \geq 0$, then in fact $\lambda_1 \geq 0$, i.e. M has nonnegative curvature operator, and all harmonic forms are parallel due to Gallot-Meyer's work.

Let $\mu_1 \leq \ldots \leq \mu_{m^2}$ denote the eigenvalues of the *Kähler curvature operator* $\mathfrak{R}_{|\mathfrak{u}(m)} \colon \mathfrak{u}(m) \to \mathfrak{u}(m)$.

Kähler manifolds

Theorem (Petersen-W, 2020)

Let (M, g) be a compact Kähler manifold of complex dimension $m \geq 3$.

If $\mu_1 + \mu_2 + \left(1 - \frac{2}{m}\right) \mu_3 > 0$, then M has the rational cohomology ring of $\mathbb{CP}^m$.

1. Classification of manifolds with nonnegative/positive
   ... bisectional curvature: Mori, Siu-Yau, Mok
   ... orthogonal bisectional curvature: Chen, Gu-Zhang
2. Proof relies on estimates for individual Hodge numbers, e.g. if

$$\mu_1 + \ldots + \mu_{m+1-p} > 0,$$

   then $h^{p,p}(M) = 1$.

3. Vanishing results for holomorphic p-forms, $h^{0,p} = 0$: Bochner, Yang, Ni-Zheng

Kähler manifolds

Theorem (Petersen-W, 2020)

Let (M, g) be a compact Kähler manifold of complex dimension m.

If $\mu_1 + \mu_2 + \left(1 - \frac{2}{m}\right)\mu_3 > 0$, then M has the rational cohomology ring of $\mathbb{CP}^m$.

4. Reduced holonomy simplifies curvature of Lichnerowicz Laplacian

$$g(\mathrm{Ric}(\varphi), \overline{\varphi}) = \sum_{\Xi_\alpha \in \mathfrak{u}(m)} \mu_\alpha |\Xi_\alpha \varphi|^2$$

5. Prove estimates

$$|\Xi_\alpha \varphi|^2 \leq c(E) \cdot |\mathring{\varphi}|^2$$

$$\sum_{\Xi_\alpha \in \mathfrak{u}(m)} |\Xi_\alpha \varphi|^2 = C(E) \cdot |\mathring{\varphi}|^2$$

for φ in each $U(m)$-irreducible module E of $\Lambda^{p,q} T^* M$

Kähler manifolds

Vanishing, Rigidity and Estimation results:

Theorem (Petersen-W, 2020)

Let (M, g) be a compact Kähler manifold of complex dimension $m \geq 3$.

1. *If* $\mu_1 + \mu_2 + \left(1 - \frac{2}{m}\right) \mu_3 > 0$, *then M has the rational cohomology ring of* $\mathbb{CP}^m$.
2. *If* $\mu_1 + \mu_2 + \left(1 - \frac{2}{m}\right) \mu_3 \geq 0$, *then all harmonic forms are parallel.*
3. *Let* $\kappa \leq 0$ *and* $D > 0$. *If* $\mu_1 + \mu_2 + \left(1 - \frac{2}{m}\right) \mu_3 \geq \kappa$ *and* $\mathrm{diam}(M) < D$, *then*

$$b_p \leq \binom{2m}{p} \exp\left(C\left(m, \kappa D^2\right) \cdot \sqrt{-\kappa D^2} \right).$$

Tachibana-type Theorems

Let (M^n, g) be a compact Riemannian manifold. Suppose (M, g) is Einstein. Then

$$\Delta \frac{1}{2} |\operatorname{Rm}|^2 = |\nabla \operatorname{Rm}|^2 + \frac{1}{2} g(\operatorname{Ric}(\operatorname{Rm}), \operatorname{Rm})$$

where

$$g(\operatorname{Ric}(\operatorname{Rm}), \operatorname{Rm}) = \sum_{\Xi_\alpha \in \mathfrak{so}(n)} \lambda_\alpha |\Xi_\alpha \operatorname{Rm}|^2$$

1. Tachibana, 1974:
   If $\lambda_1 \geq 0$, then $\nabla \operatorname{Rm} = 0$.
   If $\lambda_1 > 0$ then (M, g) has constant sectional curvature.
2. Brendle, 2010:
   If (M, g) has nonnegative isotropic curvature, then $\nabla \operatorname{Rm} = 0$.
   If (M, g) has positive isotropic curvature, then (M, g) has constant sectional curvature.

Tachibana-type Theorems

Theorem (Petersen-W, 2019)

Let (M^n, g) *be a closed, connected Einstein manifold. If*

$$\lambda_1 + \ldots + \lambda_{\lfloor \frac{n-1}{2} \rfloor} \geq 0 \text{ for } n \geq 5,$$

then the curvature tensor is parallel. Moreover, if the inequality is strict, then (M, g) *has constant sectional curvature.*

In dimension $n = 4$: If $\lambda_1 + \lambda_2 \geq 0$, then the theorem follows from work of Ni-Wu, Böhm-Wilking

Tachibana-type Theorems

Theorem (Petersen-W, 2020)

Suppose that (M, g) is a compact connected Kähler-Einstein manifold of complex dimension $m \geq 4$. If

$$\mu_1 + \ldots + \mu_{\lfloor \frac{m+1}{2} \rfloor} + \frac{1 + (-1)^m}{4} \cdot \mu_{\lfloor \frac{m+1}{2} \rfloor + 1} \geq 0,$$

then the curvature tensor is parallel.

If the inequality is strict, then (M, g) has constant holomorphic sectional curvature.

Tachibana-type theorems for Kähler manifolds with ...

... nonnegative bisectional curvature: Mori, Siu-Yau, Mok

... nonnegative orthogonal bisectional curvature: Chen, Gu-Zhang