a homomorphism of abelian groups, the *cokernel* of h is the quotient group $B/h(A)$. A sequence of groups and homomorphisms such as

$$\cdots \longrightarrow A_{n-1} \xrightarrow{h_{n-1}} A_n \xrightarrow{h_n} A_{n+1} \longrightarrow \cdots$$

is called *exact* if the kernel of each homomorphism is precisely the same as the image of the preceding homomorphism. Such exact sequences play a big role from Chapter VII on.

CHAPTER I

Two-Dimensional Manifolds

§1. Introduction

The topological concept of a surface or 2-dimensional manifold is a mathematical abstraction of the familiar concept of a surface made of paper, sheet metal, plastic, or some other thin material. A surface or 2-dimensional manifold is a topological space with the same local properties as the familiar plane of Euclidean geometry. An intelligent bug crawling on a surface could not distinguish it from a plane if he had a limited range of visibility.

The natural, higher-dimensional analog of a surface is an n-dimensional manifold, which is a topological space with the same local properties as Euclidean n-space. Because they occur frequently and have application in many other branches of mathematics, manifolds are certainly one of the most important classes of topological spaces. Although we define and give some examples of n-dimensional manifolds for any positive integer n, we devote most of this chapter to the case $n = 2$. Because there is a classification theorem for compact 2-manifolds, our knowledge of 2-dimensional manifolds is incomparably more complete than our knowledge of the higher-dimensional cases. This classification theorem gives a simple procedure for obtaining all possible compact 2-manifolds. Moreover, there are simple computable invariants which enable us to decide whether or not any two compact 2-manifolds are homeomorphic. This may be considered an ideal theorem. Much research in topology has been directed toward the development of analogous classification theorems for other situations. Unfortunately, no such theorem is known for compact 3-manifolds, and logicians have shown that we cannot even hope for such a complete result for n-manifolds, $n \geqq 4$. Nevertheless, the theory of higher-dimensional manifolds is currently a very active field of mathematical research and will probably continue to be so for a long time to come.

We shall use the material developed in this chapter later in the book.

§2. Definition and Examples of n-Manifolds

Assume n is a positive integer. An *n-dimensional manifold* is a Hausdorff space (i.e., a space that satisfies the T_2 separation axiom) such that each point has an open neighborhood homeomorphic to the open n-dimensional disc U^n $(= \{x \in \mathbf{R}^n : |x| < 1\})$. Usually we shall say "$n$-manifold" for short.

Examples

2.1. Euclidean n-space $\mathbf{R}^n$ is obviously an n-dimensional manifold. We can easily prove that the unit n-dimensional sphere

$$S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$$

is an n-manifold. For the point $x = (1, 0, \ldots, 0)$, the set $\{(x_1, \ldots, x_{n+1}) \in S^n : x_1 > 0\}$ is a neighborhood with the required properties, as we see by orthogonal projection on the hyperplane in $\mathbf{R}^{n+1}$ defined by $x_1 = 0$. For any other point $x \in S^n$, there is a rotation carrying x into the point $(1, 0, \ldots, 0)$. Such a rotation is a homeomorphism of S^n onto itself; hence, x also has the required kind of neighborhood.

2.2. If M^n is any n-dimensional manifold, then any open subset of M^n is also an n-dimensional manifold. The proof is immediate.

2.3. If M is an m-dimensional manifold and N is an n-dimensional manifold, then the product space $M \times N$ is an $(m + n)$-dimensional manifold. This follows from the fact that $U^m \times U^n$ is homeomorphic to U^{m+n}. To prove this, note that, for any positive integer k, U^k is homeomorphic to $\mathbf{R}^k$, and $\mathbf{R}^m \times \mathbf{R}^n$ is homeomorphic to $\mathbf{R}^{m+n}$.

In addition to the 2-sphere S^2, the reader can easily give examples of many other subsets of Euclidean 3-space $\mathbf{R}^3$, which are 2-manifolds, e.g., surfaces of revolution, etc.

As these examples show, an n-manifold may be either connected or disconnected, compact or noncompact. In any case, an n-manifold is always locally compact.

What is not so obvious is that a connected manifold need not satisfy the second axiom of countability (i.e., it need not have a countable base). The simplest example is the "long line."[1] Such manifolds are usually regarded as pathological, and we shall restrict our attention to manifolds with a countable base.

[1] See *General Topology* by J. L. Kelley. Princeton, N.J.: Van Nostrand, 1955. Exercise L, p. 164.

Note that in our definition we required that a manifold satisfy the Hausdorff separation axiom. We must make this requirement explicit in the definition because it is *not* a consequence of the other conditions imposed on a manifold. We leave it to the reader to construct examples of non-Hausdorff spaces, such that each point has an open neighborhood homeomorphic to U^n for $n = 1$ or 2.

§3. Orientable vs. Nonorientable Manifolds

Connected n-manifolds for $n > 1$ are divided into two kinds: orientable and nonorientable. We will try to make the distinction clear without striving for mathematical precision.

Consider the case where $n = 2$. We can prescribe in various ways an orientation for the Euclidean plane $\mathbf{R}^2$ or, more generally, for a small region in the plane. For example, we could designate which of the two possible kinds of coordinate systems in the plane is to be considered a right-handed coordinate system and which is to be considered a left-handed coordinate system. Another way would be to prescribe which direction of rotation in the plane about a point is to be considered the positive direction and which is to be considered the negative direction. Let us imagine an intelligent bug or some 2-dimensional being constrained to move in the plane; once he decides on a choice of orientation at any point in the plane, he can carry this choice with him as he moves about. If two such bugs agree on an orientation at a given point in the plane, and one of them travels on a long trip to some distant point in the plane and eventually returns to his starting point, both bugs will still agree on their choice of orientation.

Similar considerations apply to any connected 2-dimensional manifold because each point has a neighborhood homeomorphic to a neighborhood of a point in the plane. Here our two hypothetical bugs agree on a choice of orientation at a given point. It is possible, however, that after one of them returns from a long trip to some distant point on the manifold, they may find they are no longer in agreement. This phenomenon can occur even though both were meticulously careful about keeping an accurate check of the positive orientation.

The simplest example of a 2-dimensional manifold exhibiting this phenomenon is the well-known Möbius strip. As the reader probably knows, we construct a model of a Möbius strip by taking a long, narrow rectangular strip of paper and gluing the ends together with a half twist (see Figure 1.1). Mathematically, a Möbius strip is a topological space that is described as follows. Let X denote the following rectangle in the plane:

$$X = \{(x, y) \in \mathbf{R}^2 : -10 \leqq x \leqq +10,\ -1 < y < +1\}.$$

We then form a quotient space of X by identifying the points $(10, y)$ and $(-10, -y)$ for $-1 < y < +1$. Note that the two boundaries of the rectangle

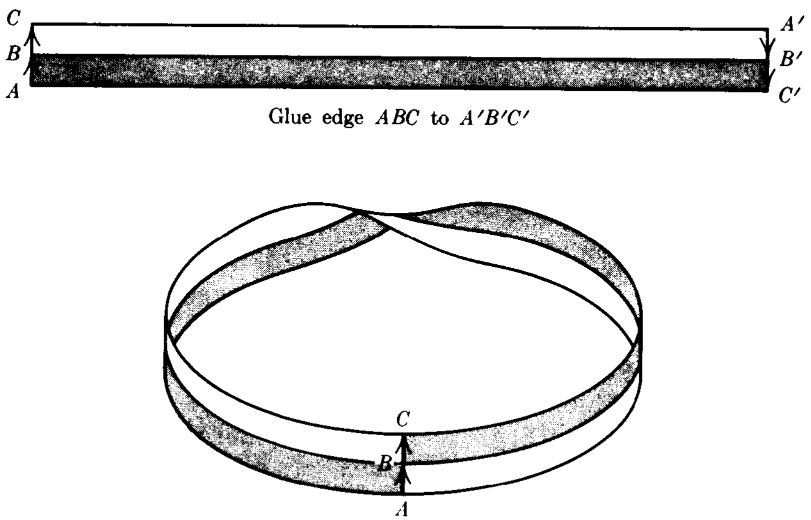


FIGURE 1.1. Constructing a Möbius strip.

corresponding to $y = +1$ and $y = -1$ were omitted. This omission is crucial; otherwise the result would not be a manifold (it would be a "manifold with boundary," a concept we will take up later in Chapter XIV). Alternatively, we could specify a certain subset of $\mathbf{R}^3$ which is homeomorphic to the quotient space just described.

However, we define the Möbius strip, the center line of the rectangular strip becomes a circle after the gluing or identification of the two ends. We leave it to the reader to verify that if our imaginary bug started out at any point on this circle with a definite choice of orientation and carried this orientation with him around the circle once, he would come back to his initial point with his original orientation reversed. We will call such a path in a manifold an *orientation-reversing* path. A closed path that does not have this property will be called an *orientation-preserving* path. For example, any closed path in the plane is orientation preserving.

A connected 2-manifold is defined to be *orientable* if every closed path is orientation preserving; a connected 2-manifold is *nonorientable* if there is at least one orientation-reversing path.

We now consider the orientability of 3-manifolds. We can specify an orientation of Euclidean 3-space or a small region thereof by designating which type of coordinate system is to be considered right handed and which type is to be considered left handed. An alternative method would be to specify which type of helix or screw thread is to be designated as right handed and which kind is to be left handed. We can now describe a closed path in a 3-manifold as *orientation preserving* or *orientation reversing*, depending on whether or not a traveler who traverses the path comes back to his initial point with his initial choice of right and left unchanged. If our universe were nonorientable, then an astronaut who made a journey along some orientation-reversing path would return to earth with the right and left sides of his body interchanged: His heart would not be on the right side of his chest, etc.

There is a 3-dimensional generalization of the Möbius strip which furnishes a particularly simple example of a nonorientable 3-manifold. Let

$$X = \{(x, y, z) \in \mathbf{R}^3 : -10 \leqq x \leqq +10,\ -1 < y < +1,\ -1 < z < +1\}.$$

Form a quotient space of X by identifying the points $(10, y, z)$ and $(-10, -y, z)$ for $-1 < y < +1$ and $-1 < z < +1$. This space may also be considered the product of an ordinary 2-dimensional Möbius strip with the open interval $\{z \in \mathbf{R} : -1 < z < +1\}$. In any case, the segment $-10 \leqq x \leqq +10$ of the x axis becomes a circle under the identification, and we leave it to the reader to convince himself that this circle is an orientation-reversing path in the resulting 3-manifold.

We will consider the analogous definitions for higher-dimensional manifolds in later chapters.

§4. Examples of Compact, Connected 2-Manifolds

To save words, from now on we shall refer to a connected 2-manifold as a *surface*. The simplest example of a compact surface is the 2-sphere S^2; another important example is the *torus*. A torus may be roughly described as any surface homeomorphic to the surface of a doughnut or of a solid ring. It may be defined more precisely as

(a) Any topological space homeomorphic to the product of two circles, $S^1 \times S^1$.
(b) Any topological space homeomorphic to the following subset of $\mathbf{R}^3$:

$$\{(x, y, z) \in \mathbf{R}^3 : [(x^2 + y^2)^{1/2} - 2]^2 + z^2 = 1\}.$$

[This is the set obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the xz plane about the z axis.]
(c) Let X denote the unit square in the plane $\mathbf{R}^2$:

$$\{(x, y) \in \mathbf{R}^2 : 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}.$$

Then, a torus is any space homeomorphic to the quotient space of X obtained by identifying opposite sides of the square X according to the following rules. The points $(0, y)$ and $(1, y)$ are to be identified for $0 \leqq y \leqq 1$, and the points $(x, 0)$ and $(x, 1)$ are to be identified for $0 \leqq x \leqq 1$.

We will find it convenient to indicate symbolically how such identifications are to be made by a diagram such as Figure 1.2. Sides that are to be identified are labeled with the same letter of the alphabet, and the identifications should be made so that the directions indicated by the arrows agree.

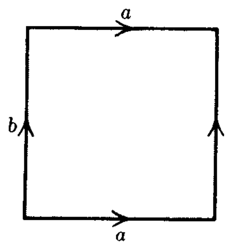


FIGURE 1.2. Construction of a torus.

We leave it to the reader to prove that the topological spaces described in (a), (b), and (c) are actually homeomorphic. The reader should also convince himself that a torus is orientable.

Our next example of a compact surface is the *real projective plane* (referred to as the *projective plane* for short). It is a compact, nonorientable surface. Because it is not homeomorphic to any subset of Euclidean 3-space, the projective plane is much more difficult to visualize than the 2-sphere or the torus.

Definition. The quotient space of the 2-sphere S^2 obtained by identifying every pair of diametrically opposite points is called a *projective plane*. We shall also refer to any space homeomorphic to this quotient space as a projective plane.

For readers who have studied projective geometry, we shall explain why this surface is called the real projective plane. Such a reader will recall that, in the study of projective plane geometry, a point has "homogeneous" coordinates (x_0, x_1, x_2), where x_0, x_1, and x_2 are real numbers, at least one of which is $\neq 0$. The term "homogeneous" means (x_0, x_1, x_2) and (x_0', x_1', x_2') represent the same point if and only if there exists a real number λ (of necessity $\neq 0$) such that

$$x_i = \lambda x_i', \quad i = 0, 1, 2.$$

If we interpret (x_0, x_1, x_2) as the ordinary Euclidean coordinates of a point in $\mathbf{R}^3$, then we see that (x_0, x_1, x_2) and (x_0', x_1', x_2') represent the same point in the projective plane if and only if they are on the same line through the origin. Thus, we may reinterpret a point of the projective plane as a line through the origin in $\mathbf{R}^3$. The next question is, how shall we topologize the set of all lines through the origin in $\mathbf{R}^3$? Perhaps th easiest way is to note that each line through the origin in $\mathbf{R}^3$ intersects the unit sphere S^2 in a pair of diametrically opposite points. This leads to the above definition.

Let $H = \{(x, y, z) \in S^2 : z \geqq 0\}$ denote the closed upper hemisphere of S^2. It is clear that, of each diametrically opposite pair of points in S^2, at least one point lies in H. If both points lie in H, then they are on the equator, which is

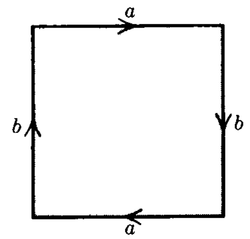


FIGURE 1.3. Construction of a projective plane from a square.

the boundary of H. Thus, we could also define the projective plane as the quotient space of H obtained by identifying diametrically opposite points on the boundary of H. As H is obviously homeomorphic to the closed unit disc E^2 in the plane,

$$E^2 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leqq 1\},$$

the quotient space of E^2 obtained by identifying diametrically opposite points on the boundary is a projective plane. For E^2 we could substitute any homeomorphic space, e.g., a square. Thus, a projective plane is obtained by identifying the opposite sides of a square as indicated in Figure 1.3. The reader should compare this with the construction of a torus in Figure 1.2.

The projective plane is easily seen to be nonorientable; in fact, it contains a subset homeomorphic to a Möbius strip.

We shall now describe how to give many additional examples of compact surfaces by forming what are called connected sums. Let S_1 and S_2 be disjoint surfaces. Their *connected sum*, denoted by $S_1 \# S_2$, is formed by cutting a small circular hole in each surface, and then gluing the two surfaces together along the boundaries of the holes. To be precise, we choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ such that D_1 and D_2 are closed discs (i.e., homeomorphic to E^2). Let S_i' denote the complement of the interior of D_i in S_i for $i = 1$ and 2. Choose a homeomorphism h of the boundary circle of D_1 onto the boundary of D_2. Then $S_1 \# S_2$ is the quotient space of $S_1' \cup S_2'$ obtained by identifying the points x and $h(x)$ for all points x in the boundary of D_1. It is clear that $S_1 \# S_2$ is a surface. It seems plausible, and can be proved rigorously, that the topological type of $S_1 \# S_2$ does not depend on the choice of the discs D_1 and D_2 or the choice of the homeomorphism h.

Examples

4.1. If S_2 is a 2-sphere, then $S_1 \# S_2$ is homeomorphic to S_1.

4.2. If S_1 and S_2 are both tori, then $S_1 \# S_2$ is homeomorphic to the surface of a block that has two holes drilled through it. (It is assumed, of course, that the holes are not so close together that their boundaries touch or intersect.)

4.3. If S_1 and S_2 are projective planes, then $S^1 \# S^2$ is a "Klein bottle," i.e., homeomorphic to the surface obtained by identifying the opposite sides of a

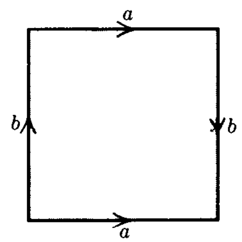


FIGURE 1.4. Construction of a Klein bottle from a square.

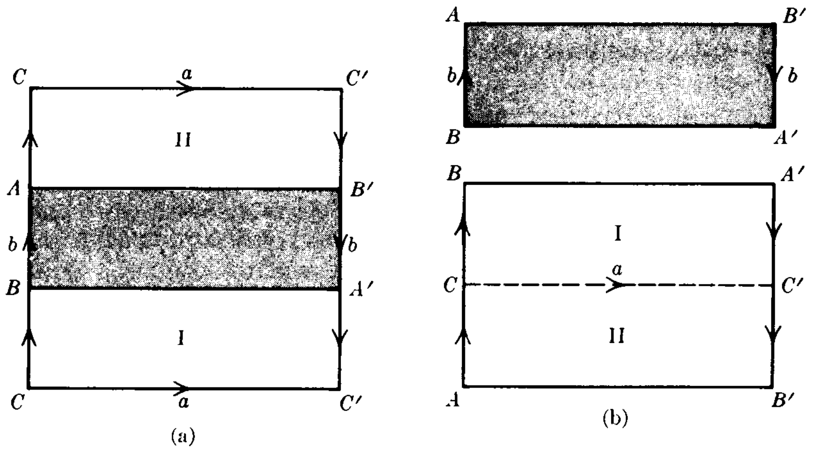


FIGURE 1.5. The Klein bottle is the union of two Möbius strips.

square as shown in Figure 1.4. We may prove this by the "cut and paste" technique, as follows. If S_i is a projective plane and D_i is a closed disc such that $D_i \subset S_i$, then S_i', the complement of the interior of D_i, is homeomorphic to a Möbius strip (including the boundary). In fact, if we think of S_i as the space obtained by identification of the diametrically opposite points on the boundary of the unit disc E^2 in $\mathbf{R}^2$, then we can choose D_i to be the image of the set $\{(x, y) \in E^2 : |y| \geqq \frac{1}{2}\}$ under the identification, and the truth of the assertion is clear. From this it follows that $S_1 \# S_2$ is obtained by gluing together two Möbius strips along their boundaries. On the other hand, Figure 1.5 shows how to cut a Klein bottle so as to obtain two Möbius strips. We cut along the lines AB' and BA'; under the identification, this cut becomes a circle.

We will now consider some properties of this operation of forming connected sums.

It is clear from our definitions that there is no distinction between $S_1 \# S_2$ and $S_2 \# S_1$; i.e., the operation is commutative. It is not difficult to see that the manifolds $(S_1 \# S_2) \# S_3$ and $S_1 \# (S_2 \# S_3)$ are homeomorphic. Thus, we see that the connected sum is a commutative, associative operation on the set of homeomorphism types of compact surfaces. Moreover, Example 4.1 shows the sphere is a unit or neutral element for this operation. We must not jump to the conclusion that the set of homeomorphism classes of compact surfaces forms a group under this operation: There are no inverses. It only forms what is called a semigroup.

The connected sum of two orientable manifolds is again orientable. On the other hand, if either S_1 or S_2 is nonorientable, then so is $S_1 \# S_2$.

§5. Statement of the Classification Theorem for Compact Surfaces

In the preceding section we have seen how examples of compact surfaces can be constructed by forming connected sums of various numbers of tori and/or projective planes. Our main theorem asserts that these examples exhaust all the possibilities. In fact, it is even a slightly stronger statement, in that we do not need to consider surfaces that are connected sums of both tori and projective planes.

Theorem 5.1. *Any compact surface is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

As preparation for the proof, we shall describe what might be called a "canonical form" for a connected sum of tori or projective planes.

Recall our description of a torus as a square with the opposite sides identified (see Figure 1.2). We can obtain an analogous description of the connected sum of two tori as follows. Represent each of the tori T_1 and T_2 as a square with opposite sides identified as shown in Figure 1.6(a). Note that all four vertices of each square are identified to a single point of the corresponding torus. To form their connected sum, we must first cut out a circular hole in each torus, and we can do this in any way that we wish. It is convenient to cut out the regions shaded in the diagrams. The boundaries of the holes are labeled c_1 and c_2, and they are to be identified as indicated by the arrows. We can also represent the complement of the holes in the two tori by the pentagons shown in Figure 1.6(b), because the indicated edge identifications imply that the two end points of the segment c_i are to be identified, $i = 1, 2$. We now identify the segments c_1 and c_2; the result is the octagon in Figure 1.6(c), in which the sides are to be identified in pairs, as indicated. Note that all eight vertices of this octagon are to be identified to a single point in $T_1 \# T_2$.

This octagon with the edges identified in pairs is our desired "canonical form" for the connected sum of two tori. By repeating this process, we can show that the connected sum of three tori is the quotient space of the 12-gon

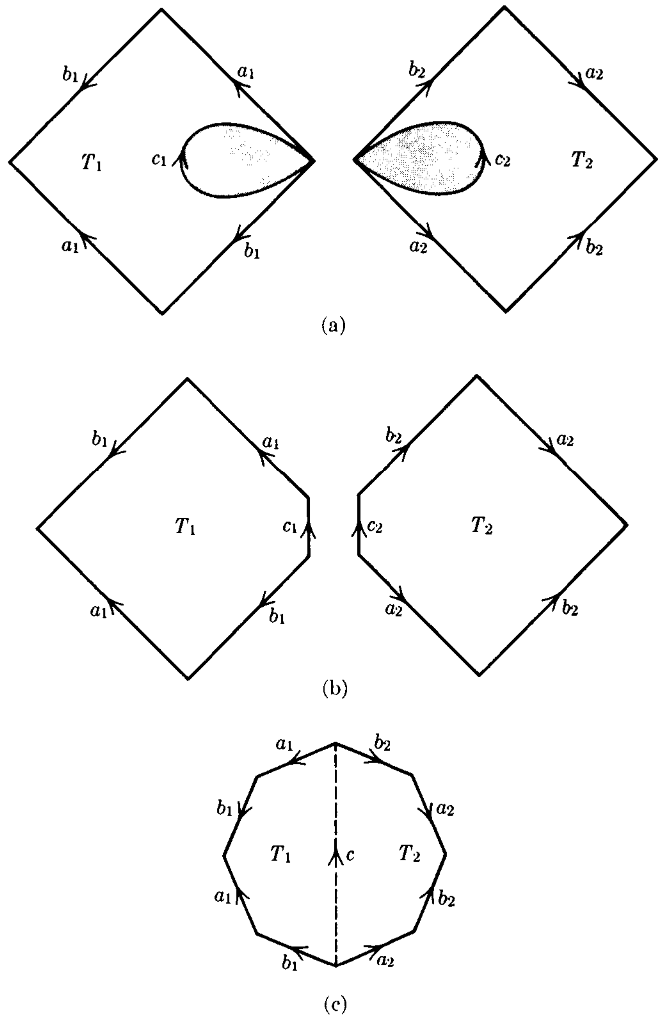


FIGURE 1.6. (a) Two disjoint tori, T_1 and T_2. (b) Disjoint tori with holes cut out. (c) After gluing together.

shown in Figure 1.7, where the edges are to be identified in pairs as indicated. It should now be clear how to prove by induction that the connected sum of n tori is homeomorphic to the quotient space of a $4n$-gon whose edges are to be identified in pairs according to a scheme, the precise description of which is left to the reader.

Next, we must consider the analogous procedure for the connected sum of projective planes. We have considered the projective plane as the quotient space of a circular disc; diametrically opposite points on the boundary are to be identified. By choosing a pair of diametrically opposite points on the

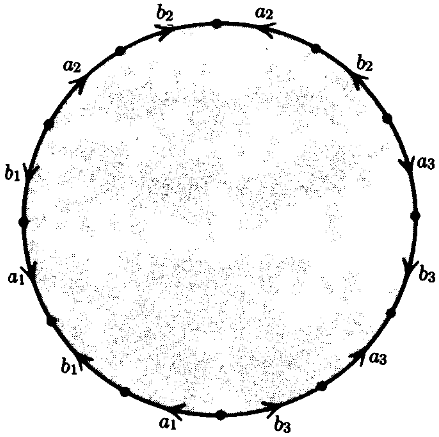


FIGURE 1.7. The connected sum of three tori is obtained by identifying the edges of a 12-gon in pairs as shown.

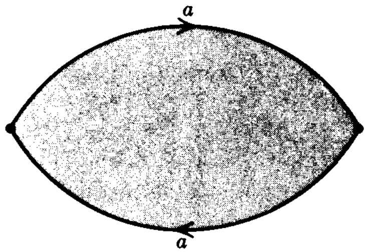


FIGURE 1.8. The projective plane is obtained by identifying opposite edges of a 2-gon.

boundary as vertices, the circumference of the disc is divided into two segments. Thus, we can regard the projective plane as obtained from a 2-gon by identification of the two edges; see Figure 1.8.

Figure 1.9 shows how to obtain a representation of the connected sum of two projective planes as a square with the edges identified in pairs. The method is basically the same as that used to obtain a representation of the connected sum of two tori as a quotient space of an octagon (Figure 1.6). By repeating this process, we see that the connected sum of three projective planes is the quotient space of a hexagon with the sides identified in pairs as indicated in Figure 1.10. By a rather obvious induction, we can prove that, for any positive integer n, the connected sum of n projective planes is the quotient space of a $2n$-gon with the sides identified in pairs according to a certain scheme. Note that all the vertices of this polygon are identified to one point.

It remains to represent the sphere as the quotient space of a polygon with the sides identified in pairs. We can do this as shown in Figure 1.11. We can think of a sphere with a zipper on it, like a purse; when the zipper is opened, the purse can be flattened out.

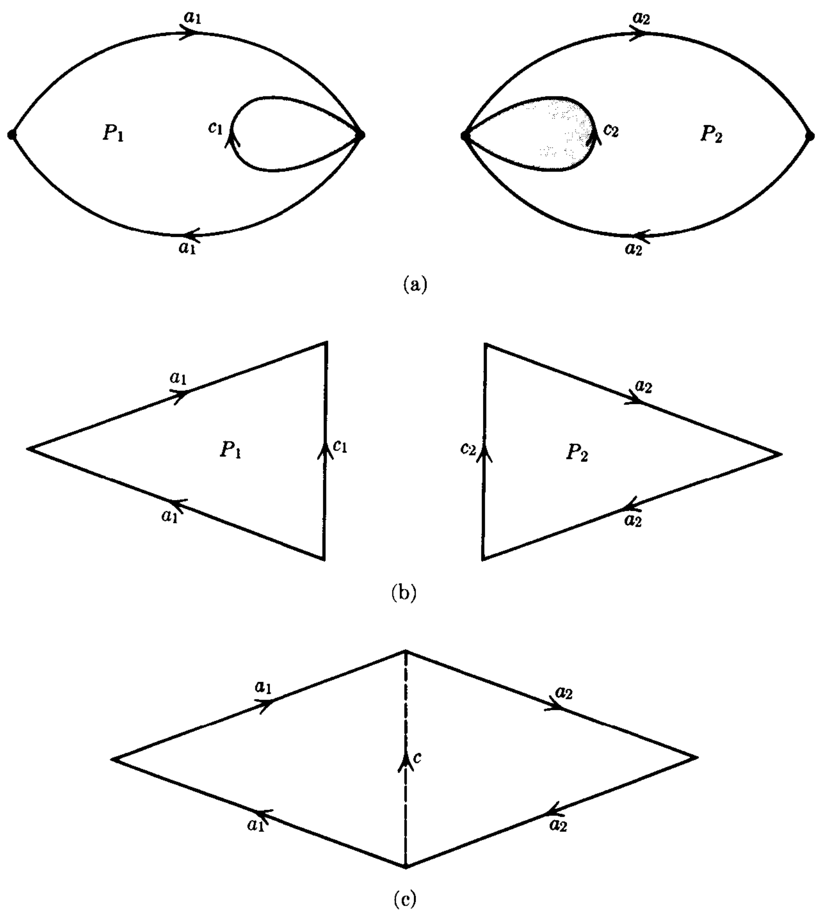


FIGURE 1.9. (a) Two disjoint projective planes, P_1 and P_2. (b) Disjoint projective planes with holes cut out. (c) After gluing together.

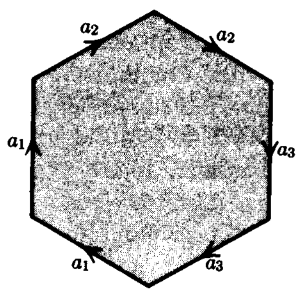


FIGURE 1.10. Construction of the connected sum of three projective planes by identifying the sides of a hexagon in pairs.

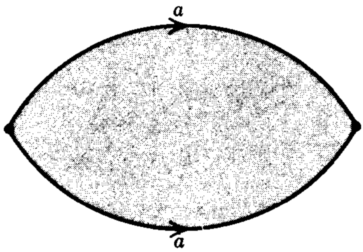


FIGURE 1.11. The sphere is a quotient space of a 2-gon with edges identified as shown.

Thus, we have shown how each of the compact surfaces mentioned in Theorem 5.1 can be considered as the quotient space of a polygon with the edges identified in pairs. We now introduce a rather obvious and convenient method of indicating precisely which paired edges are to be identified in such a polygon. Consider the diagram which indicates how the edges are identified; starting at a definite vertex, proceed around the boundary of the polygon, recording the letters assigned to the different sides in succession. If the arrow on a side points in the *same* direction that we are going around the boundary, then we write the letter for that side with no exponent (or the exponent $+1$). On the other hand, if the arrow points in the *opposite* direction, then we write the letter for that side with the exponent -1. For example, in Figures 1.7 and 1.10 the identifications are precisely indicated by the symbols

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} a_3 b_3 a_3^{-1} b_3^{-1} \qquad \text{and} \qquad a_1 a_1 a_2 a_2 a_3 a_3.$$

In each case we started at the bottom vertex of the diagram and read clockwise around the boundary. It is clear that such a symbol unambiguously describes the identifications; on the other hand, in writing the symbol corresponding to a given diagram, we can start at any vertex, and proceed either clockwise or counterclockwise around the boundary.

We summarize our results by writing the symbols corresponding to each of the surfaces mentioned in Theorem 5.1.

(a) The sphere: aa^{-1}.
(b) The connected sum of n tori:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}.$$

(c) The connected sum of n projective planes:

$$a_1 a_1 a_2 a_2 \ldots a_n a_n.$$

EXERCISES

5.1. Let P be a polygon with an even number of sides. Suppose that the sides are identified in pairs in accordance with any symbol whatsoever. Prove that the quotient space is a compact surface.

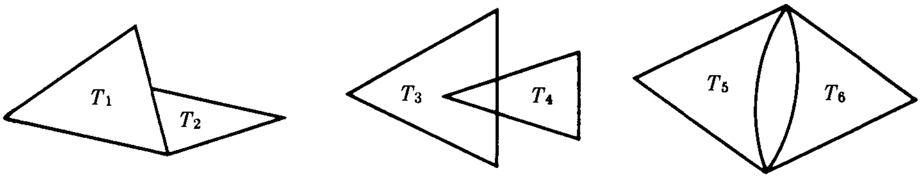


FIGURE 1.12. Some types of intersection forbidden in a triangulation.

§6. Triangulations of Compact Surfaces

To prove Theorem 5.1, we must assume that the given surface is triangulated, i.e., divided up into triangles which fit together nicely. We can easily visualize the surface of the earth divided into triangular regions, and such a subdivision is very useful in the study of compact surfaces in general.

Definition. A *triangulation* of a compact surface S consists of a finite family of closed subsets $\{T_1, T_2, \ldots, T_n\}$ that cover S, and a family of homeomorphisms $\varphi_i: T_i' \to T_i$, $i = 1, \ldots, n$, where each T_i' is a triangle in the plane $\mathbf{R}^2$ (i.e., a compact subset of $\mathbf{R}^2$ bounded by three distinct straight lines). The subsets T_i are called "triangles." The subsets of T_i that are the images of the vertices and edges of the triangle T_i' under φ_i are also called "vertices" and "edges," respectively. Finally, it is required that any two distinct triangles, T_i and T_j, either be disjoint, have a single vertex in common, or have one entire edge in common.

Perhaps the conditions in the definition are clarified by Figure 1.12, which shows three *unallowable* types of intersection of triangles.

Given any compact surface S, it seems plausible that there should exist a triangulation of S. A rigorous proof of this fact (first given by T. Radó in 1925) requires the use of a strong form of the Jordan curve theorem. Although it is not difficult, the proof is tedious, and we will not repeat it here.

We can regard a triangulated surface as having been constructed by gluing together the various triangles in a certain way, much as we put together a jigsaw puzzle or build a wall of bricks. Because two different triangles cannot have the same vertices we can specify completely a triangulation of a surface by numbering the vertices, and then listing which triples of vertices are vertices of a triangle. Such a list of triangles completely determines the surface together with the given triangulation up to homeomorphism.

Examples

6.1. The surface of an ordinary tetrahedron in Euclidean 3-space is homeomorphic to the sphere S^2; moreover, the four triangles satisfy all the conditions for a triangulation of S^2. In this case there are four vertices, and every triple

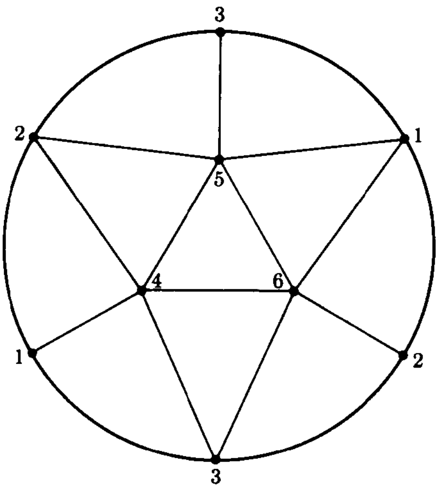


FIGURE 1.13. A triangulation of the projective plane.

of vertices is the set of vertices of a triangle. No other triangulation of any surface can have this property.

6.2. In Figure 1.13 we show a triangulation of the projective plane, considered as the space obtained by identifying diametrically opposite points on the boundary of a disc. The vertices are numbered from 1 to 6, and there are the following 10 triangles:

124	245
235	135
156	126
236	346
134	456

6.3. In Figure 1.14 we show a triangulation of a torus, regarded as a square with the opposite sides identified. There are 9 vertices, and the following 18 triangles:

124	245	235
356	361	146
457	578	658
689	649	479
187	128	289
239	379	137

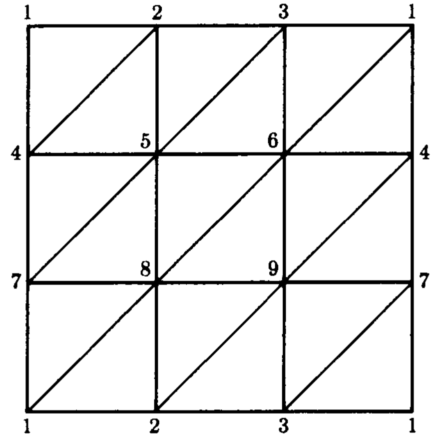


FIGURE 1.14. A triangulation of a torus.

We conclude our discussion of triangulations by noting that any triangulation of a compact surface satisfies the following two conditions:

(1) Each edge is an edge of exactly two triangles.
(2) Let v be a vertex of a triangulation. Then we may arrange the set of all triangles with v as a vertex in cyclic order, $T_0, T_1, T_2, \ldots, T_{n-1}, T_n = T_0$, such that T_i and T_{i+1} have an edge in common for $0 \leqq i \leqq n-1$.

The truth of (1) follows from the fact that each point on the edge in question must have an open neighborhood homeomorphic to the open disc U^2. If an edge were an edge of only one triangle or more than two triangles, this would not be possible. The rigorous proof of this last assertion can be given by using the concept of "The local homology groups at a point." We will take up this concept in Chapter VIII.

Condition (2) can be demonstrated as follows. The fact that the set of all the triangles with v as a vertex can be divided into several disjoint subsets, such that the triangles in each subset can be arranged in cyclic order as described, is an easy consequence of condition (1). However, if there were more than one such subset, then the requirement that v have a neighborhood homeomorphic to U^2 would be violated. This statement can also be proved by using local homology groups at a point.

§7. Proof of Theorem 5.1

Let S be a compact surface. We shall demonstrate Theorem 5.1 by proving that S is homeomorphic to a polygon with the edges identified in pairs as indicated by one of the symbols listed at the end of §5.

First step. From the discussion in the preceding section, we may assume that S is triangulated. Denote the number of triangles by n. We assert that we can number the triangles $T_1, T_2, \ldots, T_n$, so that the triangle T_i has an edge e_i in common with at least one of the triangles $T_1, \ldots, T_{i-1}$, $2 \leqq i \leqq n$. To prove this assertion, label any of the triangles T_1; for T_2 choose any triangle that has an edge in common with T_1, for T_3 choose any triangle that has an edge in common with T_1 or T_2, etc. If at any stage we could not continue this process, then we would have two sets of triangles $\{T_1, \ldots, T_k\}$, and $\{T_{k+1}, \ldots, T_n\}$ such that no triangle in the first set would have an edge or vertex in common with any triangle of the second set. But this would give a partition of S into two disjoint nonempty closed sets, contrary to the assumption that S was connected.

We now use this ordering of the triangles, $T_1, T_2, \ldots, T_n$, together with the choice of edges $e_2, e_3, \ldots, e_n$, to construct a "model" of the surface S in the Euclidean plane; this model will be a polygon whose sides are to be identified in pairs. Recall that for each triangle T_i there exists an ordinary Euclidean triangle T_i' in $\mathbf{R}^2$ and a homeomorphism φ_i of T_i' onto T_i. We can assume that the triangles $T_1', T_2', \ldots, T_n'$ are pairwise disjoint; if they are not, we can translate some of them to various other parts of the plane $\mathbf{R}^2$. Let

$$T' = \bigcup_{i=1}^{n} T_i';$$

then T' is a compact subset of $\mathbf{R}^2$. Define a map $\varphi : T' \to S$ by $\varphi | T_i' = \varphi_i$; the map φ is obviously continuous and onto. Because T' is compact and S is a Hausdorff space, φ is a closed map, and hence S has the quotient topology determined by φ. This is a rigorous mathematical statement of our intuitive idea that S is obtained by gluing the triangles $T_1, T_2, \ldots$ together along the appropriate edges.

The polygon we desire will be constructed as a quotient space of T'. Consider any of the edges e_i, $2 \leqq i \leqq n$. By assumption, e_i is an edge of the triangle T_i and one other triangle T_j, for which $1 \leqq j < i$. Therefore, $\varphi^{-1}(e_i)$ consists of an edge of the triangle T_i' and an edge of the triangle T_j'. We identify these two edges of the triangles T_i' and T_j' by identifying points which map onto the same point of e_i (speaking intuitively, we glue together the triangles T_i' and T_j'). We make these identifications for each of the edges $e_2, e_3, \ldots, e_n$. Let D denote the resulting quotient space of T'. It is clear that the map $\varphi : T' \to S$ induces a map ψ of D onto S, and that S has the quotient topology induced by ψ (because D is compact and S is Hausdorff, ψ is a closed map).

We now assert that topologically D is a closed disc. The proof depends on two facts:

(a) Let E_1 and E_2 be disjoint spaces, which topologically are closed discs (i.e., they are homeomorphic to E^2). Let A_1 and A_2 be subsets of the boundary of E_1 and E_2, respectively, which are homeomorphic to the closed interval $[0, 1]$, and let $h : A_1 \to A_2$ be a definite homeomorphism. Form a quotient space of $E_1 \cup E_2$ by identifying points that correspond under h. Then, topologically, the quotient space is also a closed disc. The reader may either take this very plausible fact for granted, or construct a proof using the type of argument given in II.8. Intuitively, it means that if we glue two discs together along a common segment of their boundaries, the result is again a disc.

(b) In forming the quotient space D of T', we may either make all the identifications at once, or make the identifications corresponding to e_2, then those corresponding to e_3, etc., in succession. This is a consequence of standard theorems about quotient spaces.

We now use these facts to prove that D is a disc as follows. T_1' and T_2' are topologically equivalent to discs. Therefore, the quotient space of $T_1' \cup T_2'$ obtained by identifying points of $\varphi^{-1}(e_2)$ is again a disc by (a). Form a quotient space of this disc and T_3' by making the identifications corresponding to the edge e_3, etc.

It is clear that S is obtained from D by identifying certain paired edges on the boundary of D.

Examples

7.1. Figure 1.15 shows an easily visualized example. The surface of a cube has been triangulated by dividing each face by a diagonal into two triangles. The resulting disc D might look like the diagram, depending, of course, on how the triangles were enumerated, and how the edges $e_2, \ldots, e_{12}$ were chosen. The edges to D that are to be identified are labeled in the usual way. At this stage, we can forget about the edges $e_2, e_3, \ldots, e_{12}$. Thus, instead of the polygon in Figure 1.15, we could work equally well with the one in Figure 1.16.

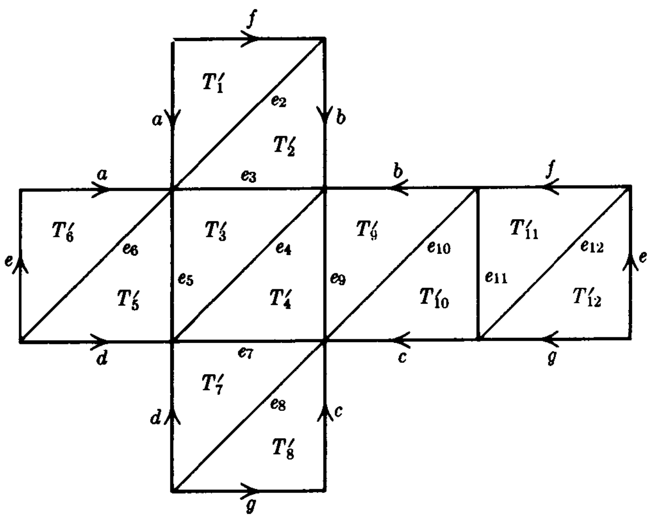


FIGURE 1.15. Example illustrating the first step of the proof of Theorem 5.1.

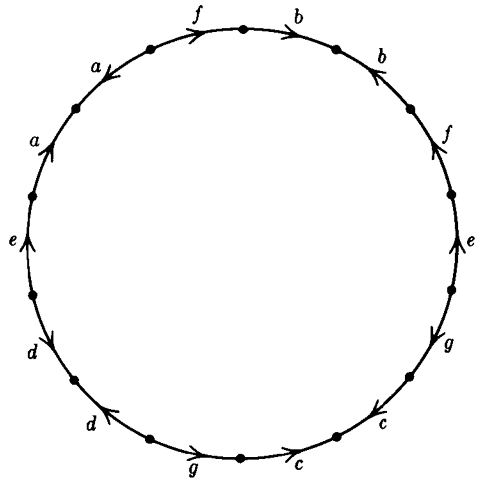


FIGURE 1.16. Simplified version of polygon shown in Figure 1.15.

EXERCISES

Carry out the above process for each of the surfaces whose triangulations are given below. (NOTE: these examples will be used later.)

7.1.

124	236	134	246
367	347	469	459
698	678	457	259
289	578	358	125
238	135		

7.2.

123	234	341	412

7.3.

123	234	345	451	512
136	246	356	416	526

7.4.

124	235	346	457	561	672
713	134	245	356	467	571
126	237				

7.5.

123	256	341	451
156	268	357	468
167	275	374	476
172	283	385	485

Second step. Elimination of adjacent edges of the first kind. We have now obtained a polygon D whose edges have to be identified in pairs to obtain the given surface S. These identifications may be indicated by the appropriate symbol; e.g., in Figure 1.16, the identifications are described by

$$aa^{-1}fbb^{-1}f^{-1}e^{-1}gcc^{-1}g^{-1}dd^{-1}e.$$

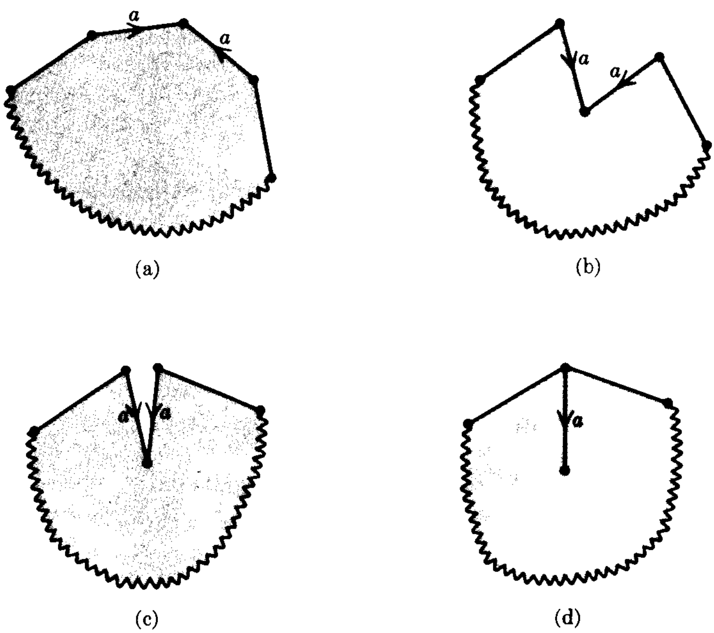


FIGURE 1.17. Elimination of an adjacent pair of edges of the first kind.

If the letter designating a certain pair of edges occurs with *both* exponents, $+1$ and -1, in the symbol, then we will call that pair of edges a pair of the *first kind*; otherwise, the pair is of the *second kind.* For example, in Figure 1.16, all seven pairs are of the first kind.

We wish to show that an adjacent pair of edges of the first kind can be eliminated, provided there are at least four edges in all. This is easily seen from the sequence of diagrams in Figure 1.17. We can continue this process until all such pairs are eliminated, or until we obtain a polygon with only two sides. In the latter case, this polygon, whose symbol will be aa or aa^{-1}, must be a projective plane or a sphere, and we have completed the proof. Otherwise, we proceed as follows.

Third step. Transformation to a polygon such that all vertices must be identified to a single vertex. Although the edges of our polygon must be identified in pairs, the vertices may be identified in sets of one, two, three, four, ... Let us call two vertices of the polygon *equivalent* if and only if they are to be identified. For example, the reader can easily verify that in Figure 1.16 there are eight different equivalence classes of vertices. Some equivalence classes contain only one vertex, whereas other classes contain two or three vertices.

Assume we have carried out step two as far as possible. We wish to prove we can transform our polygon into another polygon with all its vertices belonging to one equivalence class.

Suppose there are at least two different equivalence classes of vertices. Then,

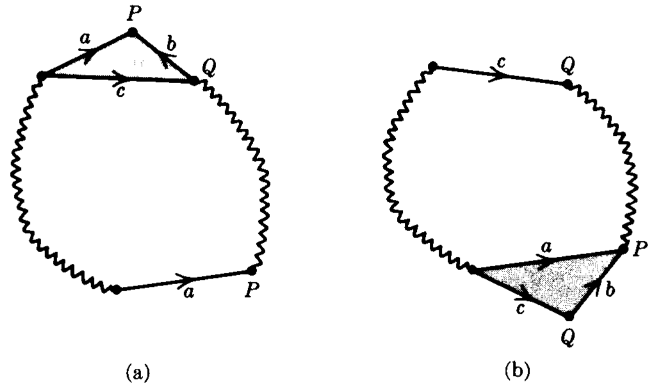


FIGURE 1.18. Third step in the proof of Theorem 5.1.

the polygon must have an adjacent pair of vertices which are nonequivalent. Label these vertices P and Q. Figure 1.18 shows how to proceed. As P and Q are nonequivalent, and we have carried out step two, it follows that sides a and b are *not* to be identified. Make a cut along the line labeled c, from the vertex labeled Q to the other vertex of the edge a (i.e., to the vertex of edge a, which is distinct from P). Then, glue the two edges labeled a together. A new polygon with one less vertex in the equivalence class of P and one more vertex in the equivalence class of Q results. If possible, perform step two again. Then carry out step three to reduce the number of vertices in the equivalence class of P still further, then do step two again. Continue alternately doing step three and step two until the equivalence class of P is eliminated entirely. If more than one equivalence class of vertices remains, we can repeat this procedure to reduce the number by 1. If we continue in this manner, we ultimately obtain a polygon such that all the vertices are to be identified to a single vertex.

Fourth step. How to make any pair of edges of the second kind adjacent. We wish to show that our surface can be transformed so that any pair of edges of the second kind are adjacent to each other. Suppose we have a pair of edges of the second kind which are nonadjacent, as in Figure 1.19(a). Cut along the

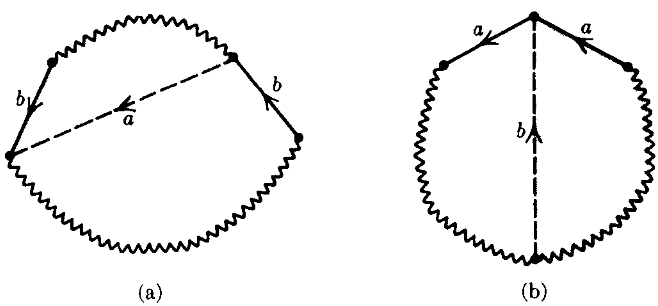


FIGURE 1.19. Fourth step in the proof of Theorem 5.1.

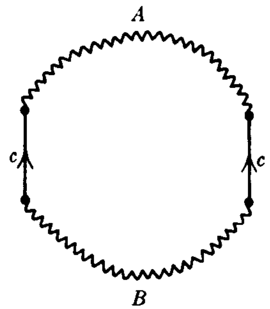


FIGURE 1.20. A pair of edges of the first kind.

dotted line labeled a and paste together along b. As shown in Figure 1.19(b), the two edges are now adjacent.

Continue this process until all pairs of edges of the second kind are adjacent. If there are no pairs of the first kind, we are finished, because the symbol of the polygon must then be of the form $a_1a_1a_2a_2\ldots a_na_n$, and hence S is the connected sum of n projective planes.

Assume to the contrary that at this stage there is at least one pair of edges of the first kind, each of which is labeled with the letter c. Then we assert that there is at least one other pair of edges of the first kind such that these two pairs separate one another; i.e., edges from the two pairs occur alternately as we proceed around the boundary of the polygon (hence, the symbol must be of the form $c\ldots d\ldots c^{-1}\ldots d^{-1}\ldots$, where the dots denote the possible occurrence of other letters).

To prove this assertion, assume that the edges labeled c are not separated by any other pair of the first kind. Then our polygon has the appearance indicated in Figure 1.20. Here A and B each designate a whole sequence of edges. The important point is that any edge in A must be identified with another edge in A, and similarly for B. No edge in A is to be identified with an edge in B. But this contradicts the fact that the initial and final vertices of either edge labeled c are to be identified, in view of step three.

Fifth step. Pairs of the first kind. Suppose, then, that we have two pairs of the first kind which separate each other as described (see Figure 1.21). We shall show that we can transform the polygon so that the four sides in question are consecutive around the perimeter of the polygon.

First, cut along c and paste together along b to obtain Figure 1.21(b). Then, cut along d and paste together along a to obtain Figure 1.21(c), as desired.

Continue this process until all pairs of the first kind are in adjacent groups of four, as $cdc^{-1}d^{-1}$ in Figure 1.21(c). If there are no pairs of the second kind, this leads to the desired result because, in that case, the symbol must be of the form

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\ldots a_nb_na_n^{-1}b_n^{-1}$$

and the surface is the connected sum of n tori.

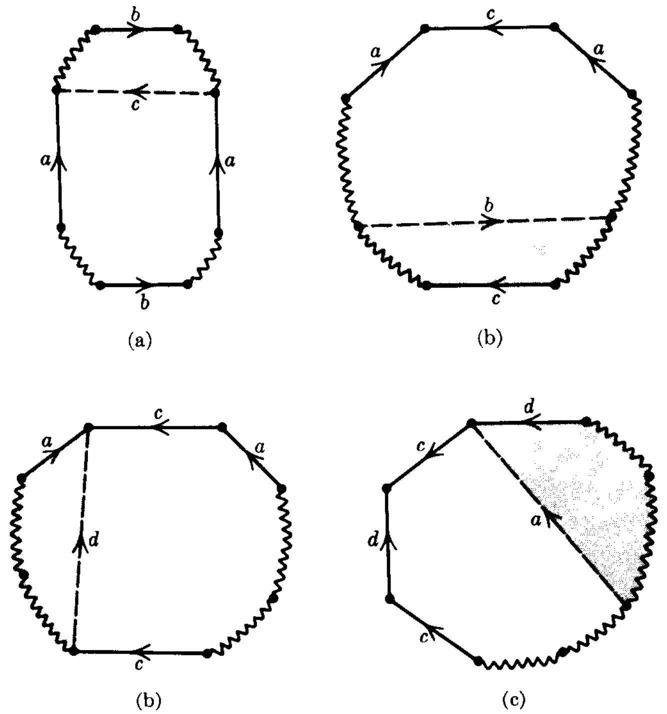


FIGURE 1.21. Fifth step in the proof of Theorem 5.1.

It remains to treat the case in which there are pairs of both the first and second kind at this stage. The key to the situation is the following rather surprising lemma:

Lemma 7.1. *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

PROOF. We have remarked that the connected sum of two projective planes is homeomorphic to a Klein bottle (see Example 4.3). Thus, we must prove that the connected sum of a projective plane and a torus is homeomorphic to the connected sum of a projective plane and a Klein bottle. To do this, it will be convenient to give an alternative construction for a connected sum of any surface S with a torus or a Klein bottle. We can represent the torus and Klein bottle as rectangles with opposite sides identified as shown in Figure 1.22. To form the connected sum, we first cut out the disc that is shaded in the diagrams, cut a similar hole in S, and glue the boundary of the hole in the torus or Klein bottle to the boundary of the hole in S. However, instead of gluing on the entire torus or Klein bottle in one step, we may do it in two stages: First, glue on the part of the torus or Klein bottle that is the image of the rectangle

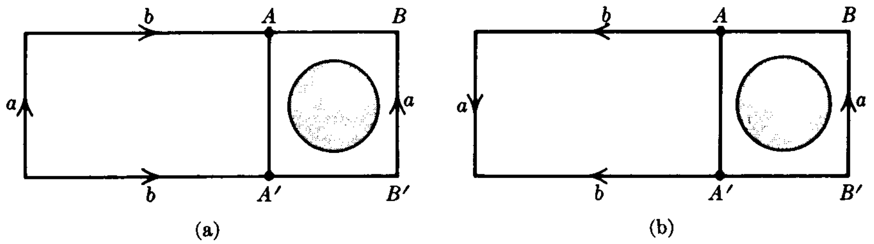


FIGURE 1.22. (a) Torus with hole. (b) Klein bottle with hole.

$ABB'A'$ under the identification, and then glue on the rest of the torus or Klein bottle. In the first stage we form the connected sum of S with an open tube or cylinder. Such an open tube or cylinder is homeomorphic to a sphere with two holes cut in it, and forming the connected sum of S with a sphere does not change anything. Thus, the space resulting from the first stage is homeomorphic to the original surface S with two holes cut in it. In the second stage we then connect the boundaries of these two holes with a tube that is the remainder of the torus or Klein bottle. The difference between the two cases depends on whether we connect the boundaries so they will have the same or opposite orientations. This is illustrated in Figure 1.23, where S is a Möbius strip.

We now assert that the two spaces shown in Figures 1.23(a) and 1.23(b) (i.e., the connected sum of a Möbius strip with a torus and a Klein bottle, respectively) are homeomorphic. To see this, imagine that we cut each of these topological spaces along the lines AB. In each case, the result is the connected sum of a rectangle and a torus, with the two ends of the rectangle to be identified with a twist, as shown in Figure 1.24. Hence, the two spaces are homeomorphic.

As stated previously, we obtain the projective plane by gluing the boundary of a disc to the boundary of a Möbius strip. As the spaces shown in Figure 1.23 are homeomorphic, so are the spaces obtained by gluing a disc on the boundary of each. Thus, the connected sum of a projective plane and a torus is homeomorphic to the connected sum of a projective plane and a Klein bottle, as was to be proved. Q.E.D.

It should be clear that this lemma takes care of the remaining case. Assume that after the fifth step has been completed, the polygon has m pairs ($m > 0$) of the second kind such that the two edges of each pair are adjacent, and n quadruples ($n > 0$) of sides, each quadruple consisting of two pairs of the first kind which separate each other. Then, the surface is the connected sum of m projective planes and n tori, which by the lemma is homeomorphic to the connected sum of $m + 2n$ projective planes. This completes the proof of Theorem 5.1.

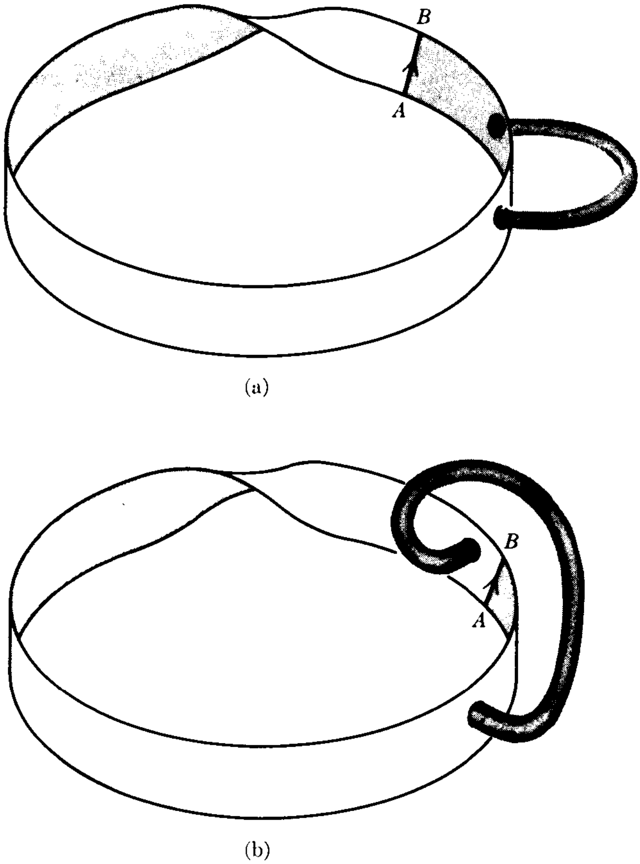


FIGURE 1.23. (a) Connected sum of a Möbius strip and a torus. (b) Connected sum of a Möbius strip and a Klein bottle.

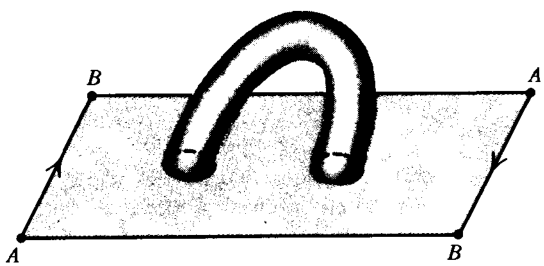


FIGURE 1.24. The result of cutting the spaces shown in Figure 1.23 along the line AB.

EXERCISES

7.6. Carry out each of the above steps for the examples given in Exercises 7.1–7.5.

It is clear that we can also work the process described above backwards; whenever there are three pairs of the second kind, we can replace them by one pair of the second kind and two pairs of the first kind. Alternatively, we can apply Lemma 7.1 to any connected sum of which three or more of the summands are projective planes. The following alternative form of Theorem 5.1, which may be preferable in some cases, results.

Theorem 7.2. *Any compact, orientable surface is homeomorphic to a sphere or a connected sum of tori. Any compact, nonorientable surface is homeomorphic to the connected sum of either a projective plane or Klein bottle and a compact, orientable surface.*

§8. The Euler Characteristic of a Surface

Although we have shown that any compact surface is homeomorphic to a sphere, a sum of tori, or a sum of projective planes, we do not know that all these are topologically different. It is conceivable that there exist positive integers m and n, $m \neq n$, such that the sum of m tori is homeomorphic to the sum of n tori. To show that this cannot happen, we introduce a numerical invariant called the *Euler characteristic.*

First, we define the Euler characteristic of a triangulated surface. Let M be a compact surface with triangulation $\{T_1, \ldots, T_n\}$. Let

$$v = \text{total number of vertices of } M,$$

$$e = \text{total number of edges of } M,$$

$$t = \text{total number of triangles (in this case, } t = n).$$

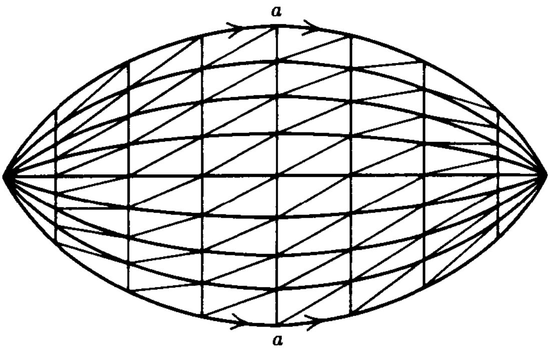
Then,

$$\chi(M) = v - e + t$$

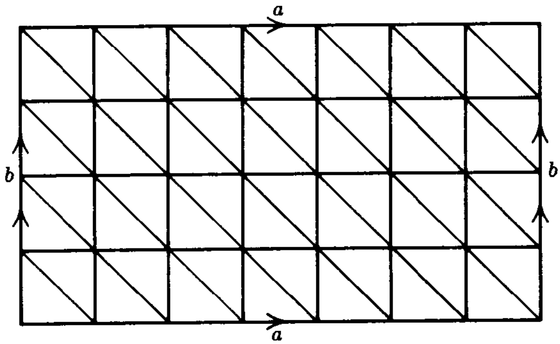
is called the *Euler characteristic* of M.

Example

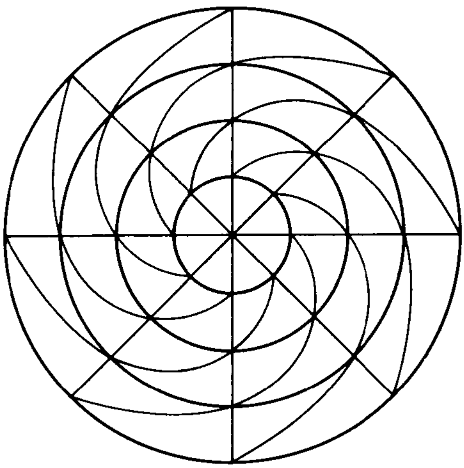
8.1. Figure 1.25 suggests uniform methods of triangulating the sphere, torus, and projective plane so that we may make the number of triangles as large as we please. Using such triangulations, the reader should verify that the Euler characteristics of the sphere, torus, and projective plane are 2, 0, and 1, respectively. He should also verify that the Euler characteristics are independent of the number of vertical and horizontal dividing lines in the diagrams



(a)



(b)



(c)

FIGURE 1.25. Computing the Euler characteristic from a triangulation. (a) Sphere. (b) Torus. (c) Projective plane.

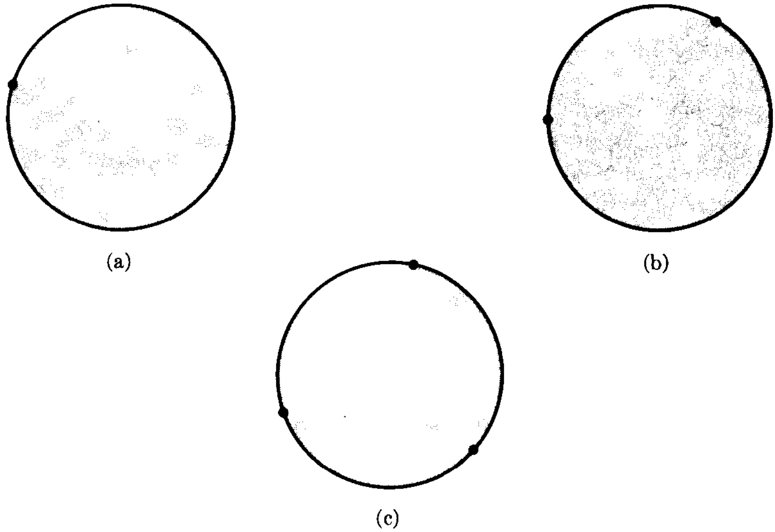


FIGURE 1.26. (a) 1-gon. (b) A 2-gon. (c) A 3-gon.

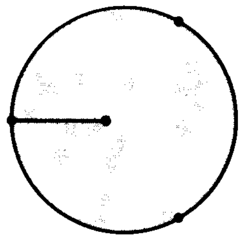


FIGURE 1.27. An allowable kind of edge.

for the sphere and torus, and of the number of radial lines or concentric circles in the case of the diagram for the projective plane.

Consideration of these and other examples suggests that $\chi(M)$ depends only on M, not on the triangulation chosen. We wish to suggest a method of proving this. To do this, we shall allow subdivisions of M into arbitrary polygons, not just triangles. These polygons may have any number n of sides and vertices, $n \geqq 1$ (see Figure 1.26). We shall also allow for the possibility of edges that do not subdivide a region, as in Figure 1.27. In any case, the interior of each polygonal region is required to be homeomorphic to an open disc, and each edge is required to be homeomorphic to an open interval of the real line, once the vertices are removed (the closure of each edge shall be homeomorphic to a closed interval or a circle). Finally, the number of vertices, edges, and polygonal regions will be finite. As before, we define the Euler characteristic of such a subdivision of a compact surface M to be

$$\chi(M) = (\text{No. of vertices}) - (\text{No. of edges}) + (\text{No. of regions}).$$

It is now easily shown that the Euler characteristic is invariant under the following processes:

(a) Subdividing an edge by adding a new vertex at an interior point (or, inversely, if only two edges meet at a given vertex, we can amalgamate the two edges into one and eliminate the vertex).
(b) Subdividing an n-gon, $n \geqq 1$, by connecting two of the vertices by a new edge (or, inversely, amalgamating two regions into one by removing an edge).
(c) Introducing a new edge and vertex running into a region, as shown in Figure 1.27 (or, inversely, eliminating such an edge and vertex).

The invariance of the Euler characteristic would now follow if it could be shown that we could get from any one triangulation (or subdivision) to any other by a finite sequence of "moves" of types (a), (b), and (c). Suppose we have two triangulations

$$\mathscr{T} = \{T_1, T_2, \ldots, T_m\},$$
$$\mathscr{T}' = \{T_1', T_2', \ldots, T_n'\}$$

of a given surface. If the intersection of any edge of the triangulation $\mathscr{T}$ with any edge of the triangulation $\mathscr{T}'$ consists of a finite number of points and a finite number of closed intervals, then it is easily seen that we can get from the triangulation $\mathscr{T}$ to the triangulation $\mathscr{T}'$ in a finite number of such moves; the details are left to the reader. However, it may happen that an edge of $\mathscr{T}$ intersects an edge of $\mathscr{T}'$ in an infinite number of points, like the following two curves in the xy plane:

$$\{(x, y) : y = 0 \quad \text{and} \quad -1 \leqq x \leqq +1\},$$
$$\left\{(x, y) : y = x \sin\frac{1}{x} \quad \text{and} \quad 0 < |x| \leqq 1\right\} \cup \{(0,0)\}.$$

If this is the case, it is clearly impossible to get from the triangulation $\mathscr{T}$ to the triangulation $\mathscr{T}'$ by any finite number of moves. It appears plausible that we could always avoid such a situation by "moving" one of the edges slightly. This is true and can be proved rigorously. However, we do not attempt such a proof here for several reasons: (a) The details are tedious and involved. (b) In Chapter IX we will define the Euler characteristic for a more general class of topological spaces and prove its invariance by means of homology theory. In these more general circumstances, the type of proof we have suggested here is not possible. (c) We will use the Euler characteristic to distinguish between compact surfaces. We will achieve this purpose with complete rigor in later chapters by the use of the fundamental group and by use of homology groups.

Proposition 8.1. *Let S_1 and S_2 be compact surfaces. The Euler characteristics of S_1 and S_2 and their connected sum, $S_1 \# S_2$, are related by the formula*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

PROOF. The proof is very simple; assume S_1 and S_2 are triangulated. Form their connected sum by removing from each the interior of a triangle, and then identifying edges and vertices of the boundaries of the removed triangles. The formula then follows by counting vertices, edges, and triangles before and after the formation of the connected sum. Q.E.D.

Using this proposition, and an obvious induction, starting from the known results for the sphere, torus, and projective plane, we obtain the following values for the Euler characteristics of the various possible compact surfaces:

Surface	*Euler characteristic*
Sphere	2
Connected sum of n tori	$2 - 2n$
Connected sum of n projective planes	$2 - n$
Connected sum of projective plane and n tori	$1 - 2n$
Connected sum of Klein bottle and n tori	$-2n$

Note that the Euler characteristic of an orientable surface is always even, whereas for a nonorientable surface it may be either odd or even.

Assuming the topological invariance of the Euler characteristic and Theorem 5.1, we have the following important result:

Theorem 8.2. *Let S_1 and S_2 be compact surfaces. Then, S_1 and S_2 are homeomorphic if and only if their Euler characteristics are equal and both are orientable or both are nonorientable.*

This is a topological theorem par excellence; it reduces the classification problem for compact surfaces to the determination of the orientability and Euler characteristic, both problems usually readily soluble. Moreover, Theorem 5.1 makes clear what are all possible compact surfaces.

Such a complete classification of any class of topological spaces is very rare. No corresponding theorem is known for compact 3-manifolds, and for 4-manifolds it has been proven (roughly speaking) that no such result is possible.

We close this section by giving some standard terminology. A surface that is the connected sum of n tori or n projective planes is said to be of *genus* n, whereas a sphere is of *genus* 0. The following relation holds between the genus g and the Euler characteristic χ of a compact surface:

$$g = \begin{cases} \frac{1}{2}(2 - \chi) & \text{in the orientable case} \\ 2 - \chi & \text{in the nonorientable case.} \end{cases}$$

EXERCISES

8.1. For over 2000 years it has been known that there are only five regular polyhedra, namely, the regular tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Prove this by considering subdivisions of the sphere into n-gons (n fixed) such that exactly m edges meet at each vertex (m fixed, $m, n \geqq 3$). Use the fact that $\chi(S^2) = 2$.

8.2. For any *triangulation* of a compact surface, show that

$$3t = 2e,$$
$$e = 3(v - \chi),$$
$$v \geqq \tfrac{1}{2}(7 + \sqrt{49 - 24\chi}).$$

In the case of the sphere, projective plane, and torus, what are the minimum values of the numbers v, e, and t? (Here, t, e, and v denote the number of triangles, edges, and vertices, respectively.)

8.3. In how many pieces do n great circles, no three of which pass through a common point, dissect a sphere?

8.4. (a) The sides of a regular octagon are identified in pairs in such a way as to obtain a compact surface. Prove that the Euler characteristic of this surface is $\geqq -2$.
(b) Prove that any surface (orientable or nonorientable) of Euler characteristic $\geqq -2$ can be obtained by suitably identifying in pairs the sides of a regular octagon.

8.5. Prove that it is not possible to subdivide the surface of a sphere into regions, each of which has six sides (i.e., it is a hexagon) and such that distinct regions have no more than one side in common.

8.6. Let S_1 be a surface that is the sum of m tori, $m \geqq 1$, and let S_2 be a surface that is the sum of n projective planes, $n \geqq 1$. Suppose two holes are cut in each of these surfaces, and the two surfaces are then glued together along the boundaries of the holes. What surface is obtained by this process?

8.7. What surface is represented by a regular 10-gon with edges identified in pairs, as indicated by the symbol $abcdec^{-1}da^{-1}b^{-1}e^{-1}$? (HINT: How are the vertices identified around the boundary?)

8.8. What surface is represented by a $2n$-gon with the edges identified in pairs according to the symbol

$$a_1a_2 \ldots a_na_1^{-1}a_2^{-1} \ldots a_{n-1}^{-1}a_n?$$

8.9. What surface is represented by a $2n$-gon with the edges identified in pairs according to the symbol

$$a_1a_2 \ldots a_na_1^{-1}a_2^{-1} \ldots a_{n-1}^{-1}a_n^{-1}?$$

(HINT: The cases where n is odd and where n is even are different.)

Remark: The results of Exercises 8.8 and 8.9 together give an alternative "normal form" for the representation of a compact surface as a quotient space of polygon.

NOTES

Definition of the connected sum of two manifolds

The definition of the connected sum given in §4 is adequate for 2-dimensional manifolds, but more care is necessary when we define the connected sum of two orientable n-manifolds for $n > 2$. We must worry about whether the homeomorphism h in our definition preserves or reverses orientations. The essential reason for this difference is that any orientable surface admits an orientation-reversing self-homeomorphism, whereas there exist orientable manifolds in higher dimensions which do not admit such a self-homeomorphism. Seifert and Threlfall ([6], pp. 290–291) give an example of a 3-dimensional manifold with this property. The complex projective plane is a 4-dimensional manifold having the property in question.

Triangulation of manifolds

In the early days of topology, it was apparently taken for granted that all surfaces and all higher-dimensional manifolds could be triangulated. The first rigorous proof that surfaces can be triangulated was published by Tibor Radó in a paper on Riemann surfaces [7]. Radó pointed out the necessity of assuming the surface has a countable basis for its topology and gave an example (due to Prüfer) of a surface that does not have such a countable basis. Radó's proof, given in Chapter I of the text by Ahlfors and Sario [1], makes essential use of a strong form of the Jordan Curve Theorem. The triangulability of 3-manifolds was proved by E. Moise (Affine Structures in 3-manifolds, V: The triangulation theorem and Hauptvermutung. *Ann. Math.* **56** (1952), 96–114).

Recent results of A. Casson and M. Freedman show that some 4-dimensional manifolds cannot be triangulated.

Models of nonorientable surfaces in Euclidean 3-space

No closed subset of Euclidean n-space is homeomorphic to a nonorientable $(n-1)$-manifold. This result, first proved by the Dutch mathematician L.E.J. Brouwer in 1912, can now be proved as an easy corollary of some general theorems of homology theory. This fact seriously hampers the development of our geometric intuition regarding compact, nonorientable surfaces, since they cannot be imbedded homeomorphically in Euclidean 3-space. However, it is possible to construct models of such surfaces in Euclidean 3-space provided we allow "singularities" or "self-intersections." We can even construct a mathematical theory of such models by considering the concept of *immersion* of manifolds. We say that a continuous map f of a compact n-manifold M^n into m-dimensional Euclidean space $\mathbf{R}^m$ is a *topological immersion* if each point of M^n has a neighborhood mapped homeomorphically onto its image by f. (The definition of a *differentiable immersion* is analogous; f is required to be differentiable and have a Jacobian everywhere of maximal rank.) The usual model of a Klein bottle in $\mathbf{R}^3$ is an immersion of the Klein bottle in 3-space. Werner Boy, in his thesis at the University of Göttingen in 1901 [Über die Abbildung der projektiven Ebene auf eine im Endlichen geschlossene singularitätenfreie Fläche. *Nach. Königl. Gesell. Wiss. Göttingen* (*Math. Phys. Kl.*), 1901, pp. 20–33. See also *Math. Annal.* **57** (1903), 173–184], constructed immersions of the projective plane in $\mathbf{R}^3$. One of the immersions given by Boy is reproduced in Hilbert and Cohn-Vossen [3]. Since any compact, nonorientable surface is homeomorphic to the connected sum of an orientable surface and a projective plane or a Klein bottle, it is now easy to construct immersions of the remaining compact, nonorientable surfaces in $\mathbf{R}^3$.

The usual immersion of the Klein bottle in $\mathbf{R}^3$ is much nicer than any of the immersions of the projective plane given by Boy. The set of singular points for the immersion of the Klein bottle consists of a circle of double points, whereas the set of singular points for Boy's immersions of the projective plane is much more complicated. This raises the question, does there exist an immersion of the projective plane in $\mathbf{R}^3$ such that the set of singular points consists of disjoint circles of double points? The answer to this question is negative, at least in the case of differentiable immersions; for the proof, see the two papers by T. Banchoff in Proceedings of the American Mathematical Society published in 1974 (**46**, 402–413).

For further information on the immersion of compact surfaces in $\mathbf{R}^3$, see the interesting article entitled "Turning a Surface Inside Out" by Anthony Phillips in *Scientific American* published in 1966 (**214**, 112–120).

Bibliographical notes

The first proof of the classification theorem for compact surfaces is ascribed by some to H. R. Brahana (*Ann. Math.* **23** (1922), 144–68). However, Seifert and Threlfall ([6], p. 322), attribute it to Dehn and Heegard and do not even list Brahana's paper in their bibliography. During the 19th century several mathematicians worked on the classification of surfaces, especially at the time of Riemann and afterword. The nonexistence of any algorithm for the classification of compact triangulable 4-manifolds is a result of the Russian mathematician A. A. Markov (*Proc. Int. Cong. Mathematicians*, 1958, pp. 300–306). For the use of the Euler characteristic to prove the 5-color theorem for maps, see R. Courant and H. Robbins, *What Is Mathematics*? (Oxford University Press, New York, 1941, pp. 264–267). We also refer the student to excellent drawings in the books by Cairns ([2], p. 28), and Hilbert and Cohn-Vossen ([3], p. 265), illustrating how the connected sum of two or three tori can be cut open to obtain a polygon whose opposite edges are to be identified in pairs.

References

Books

1. L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, Princeton, N.J., 1960, Chapter I.
2. S. S. Cairns, *Topology*, Ronald, New York, 1961, Chapter II.
3. D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, Dover, New York, 1944. Chapter VI. There is also an English translation by P. Nemenyi entitled *Geometry and the Imagination*, Chelsea, New York, 1956.
4. B. Kerekjarto, *Vorlesungen über Topologie*, J. Springer, Berlin, 1923, Chapters IV and V.
5. K. Reidemeister, *Einführung in die kombinatorische Topologie*, Friedr. Vieweg & Sohn, Braunschweig, 1932, Chapter V.
6. H. Seifert and W. Threlfall, *A Textbook of Topology*, Academic Press, New York, 1980, Chapter 6.

Papers

7. T. Radó, Über den Begriff der Riemannschen Fläche. *Acta Litt. Sci. Szeged.* **2** (1925), 101–121.
8. I. Richards, On the classification of noncompact surfaces, *Trans. Am. Math. Soc.* **106** (1963), 259–269.
9. P. H. Doyle and D. A. Moran. A short proof that compact 2-manifolds can be triangulated, *Invent. Math.* **5** (1968), 160–162.