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The stable moduli space of Riemann surfaces: Mumford's conjecture

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Abstract

D. Mumford conjectured in [33] that the rational cohomology of the stable moduli space of Riemann surfaces is a polynomial algebra generated by certain classes κ_i of dimension $2i$. For the purpose of calculating rational cohomology, one may replace the stable moduli space of Riemann surfaces by $B\Gamma_\infty$, where Γ_∞ is the group of isotopy classes of automorphisms of a smooth oriented connected surface of "large" genus. Tillmann's theorem [44] that the plus construction makes $B\Gamma_\infty$ into an infinite loop space led to a stable homotopy version of Mumford's conjecture, stronger than the original [24]. We prove the stronger version, relying on Harer's stability theorem [17], Vassiliev's theorem concerning spaces of functions with moderate singularities [46], [45] and methods from homotopy theory.

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1. Introduction: Results and methods

1.1. *Main result.* Let $F = F_{g,b}$ be a smooth, compact, connected and oriented surface of genus $g > 1$ with $b \geq 0$ boundary circles. Let $\mathscr{H}(F)$ be the space of hyperbolic metrics on F with geodesic boundary and such that each boundary circle has unit length. The topological group $\mathrm{Diff}(F)$ of orientation preserving diffeomorphisms $F \to F$ which restrict to the identity on the boundary acts on $\mathscr{H}(F)$ by pulling back metrics. The orbit space

$$\mathscr{M}(F) = \mathscr{H}(F)\big/\mathrm{Diff}(F)$$

is the (hyperbolic model of the) moduli space of Riemann surfaces of topological type F.

The connected component $\mathrm{Diff}_1(F)$ of the identity acts freely on $\mathscr{H}(F)$ with orbit space $\mathscr{T}(F)$, the Teichmüller space. The projection from $\mathscr{H}(F)$ to $\mathscr{T}(F)$ is a principal Diff_1-bundle [7], [8]. Since $\mathscr{H}(F)$ is contractible and $\mathscr{T}(F) \cong \mathbb{R}^{6g-6+2b}$, the subgroup $\mathrm{Diff}_1(F)$ must be contractible. Hence the mapping class group $\Gamma_{g,b} = \pi_0\mathrm{Diff}(F)$ is homotopy equivalent to the full group $\mathrm{Diff}(F)$, and $B\Gamma_{g,b} \simeq B\mathrm{Diff}(F)$.

When $b > 0$ the action of $\Gamma_{g,b}$ on $\mathscr{T}(F)$ is free so that $B\Gamma_{g,b} \simeq \mathscr{M}(F)$. If $b = 0$ the action of $\Gamma_{g,b}$ on $\mathscr{T}(F)$ has finite isotropy groups and $\mathscr{M}(F)$ has singularities. In this case

$$B\Gamma_{g,b} \simeq (E\Gamma_{g,b} \times \mathscr{T}(F))/\Gamma_{g,b}$$

and the projection $B\Gamma_{g,b} \to \mathscr{M}(F)$ is only a rational homology equivalence.

For $b > 0$, the standard homomorphisms

$$\Gamma_{g,b} \;\to\; \Gamma_{g+1,b}\,, \qquad \Gamma_{g,b} \;\to\; \Gamma_{g,b-1} \tag{1.1}$$

yield maps of classifying spaces that induce isomorphisms in integral cohomology in degrees less than $g/2 - 1$ by the stability theorems of Harer [17] and Ivanov [20]. We let $B\Gamma_{\infty,b}$ denote the mapping telescope or homotopy colimit of

$$B\Gamma_{g,b} \longrightarrow B\Gamma_{g+1,b} \longrightarrow B\Gamma_{g+2,b} \longrightarrow \cdots \;\; .$$

Then $H^*(B\Gamma_{\infty,b};\mathbb{Z}) \cong H^*(B\Gamma_{g,b};\mathbb{Z})$ for $* < g/2 - 1$, and in the same range the cohomology groups are independent of b.

The mapping class groups $\Gamma_{g,b}$ are perfect for $g > 2$ and so we may apply Quillen's plus construction to their classifying spaces. By the above, the resulting homotopy type is independent of b when $g = \infty$; we write

$$B\Gamma_\infty^+ = B\Gamma_{\infty,b}^+\,.$$

The main result from [44] asserts that $\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space, so that homotopy classes of maps to it form the degree 0 part of a generalized cohomology theory. Our main theorem identifies this cohomology theory.

Let $G(d,n)$ denote the Grassmann manifold of oriented d-dimensional subspaces of $\mathbb{R}^{d+n}$, and let $U_{d,n}$ and $U_{d,n}^\perp$ be the two canonical vector bundles on $G(d,n)$ of dimension d and n, respectively. The restriction

$$U_{d,n+1}^\perp | G(d,n)$$

is the direct sum of $U_{d,n}^\perp$ and a trivialized real line bundle. This yields an inclusion of their associated Thom spaces,

$$S^1 \wedge \mathrm{Th}\,(U_{d,n}^\perp) \longrightarrow \mathrm{Th}\,(U_{d,n+1}^\perp)\,,$$

and hence a sequence of maps (in fact cofibrations)

$$\cdots \to \Omega^{n+d}\mathrm{Th}\,(U_{d,n}^\perp) \to \Omega^{n+1+d}\mathrm{Th}\,(U_{d,n+1}^\perp) \to \cdots$$

with colimit

$$\Omega^\infty \mathbf{hV} \;=\; \mathrm{colim}_n\, \Omega^{n+d}\mathrm{Th}\,(U_{d,n}^\perp). \tag{1.2}$$

For $d = 2$, the spaces $G(d, n)$ approximate the complex projective spaces, and

$$\Omega^\infty \mathbf{hV} \simeq \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} := \mathrm{colim}_n\, \Omega^{2n+2}\mathrm{Th}\,(L_n^\perp)$$

where $L_n^\perp$ is the complex n-plane bundle on $\mathbb{C}P^n$ which is complementary to the tautological line bundle L_n.

There is a map α_∞ from $\mathbb{Z} \times B\Gamma^+_\infty$ to $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ constructed and examined in considerable detail in [24]. Our main result is the following theorem conjectured in [24]:

THEOREM 1.1. *The map* $\alpha_\infty : \mathbb{Z} \times B\Gamma^+_\infty \longrightarrow \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ *is a homotopy equivalence.*

Since α_∞ is an infinite loop map by [24], the theorem identifies the generalized cohomology theory determined by $\mathbb{Z} \times B\Gamma^+_\infty$ to be the one associated with the spectrum $\mathbb{C}\mathbf{P}^\infty_{-1}$. To see that Theorem 1.1 verifies Mumford's conjecture we consider the homotopy fibration sequence of [37],

$$\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} \xrightarrow{\ \omega\ } \Omega^\infty S^\infty(\mathbb{C}P^\infty_+) \xrightarrow{\ \partial\ } \Omega^{\infty+1} S^\infty \tag{1.3}$$

where the subscript $+$ denotes an added disjoint base point. The homotopy groups of $\Omega^{\infty+1}S^\infty$ are equal to the stable homotopy groups of spheres, up to a shift of one, and are therefore finite. Thus $H^*(\omega; \mathbb{Q})$ is an isomorphism. The canonical complex line bundle over $\mathbb{C}P^\infty$, considered as a map from $\mathbb{C}P^\infty$ to $\{1\} \times B\mathrm{U}$, induces via Bott periodicity a map

$$L : \Omega^\infty S^\infty(\mathbb{C}\mathbf{P}^\infty_+) \longrightarrow \mathbb{Z} \times B\mathrm{U},$$

and $H^*(L; \mathbb{Q})$ is an isomorphism. Thus we have isomorphisms

$$H^*(\mathbb{Z} \times B\Gamma^+_\infty; \mathbb{Q}) \;\cong\; H^*(\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}; \mathbb{Q}) \;\cong\; H^*(\mathbb{Z} \times B\mathrm{U}; \mathbb{Q})\,.$$

Since Quillen's plus construction leaves cohomology undisturbed this yields Mumford's conjecture:

$$H^*(B\Gamma_\infty; \mathbb{Q}) \;\cong\; H^*(B\mathrm{U}; \mathbb{Q}) \;\cong\; \mathbb{Q}[\kappa_1, \kappa_2, \dots]\,.$$

Miller, Morita and Mumford [26], [31], [32], [33] defined the classes κ_i in $H^{2i}(B\Gamma_\infty; \mathbb{Q})$ by integration (Umkehr) of the $(i+1)$-th power of the tangential Euler class in the universal smooth $F_{g,b}$-bundles. In the above setting $\kappa_i = \alpha^*_\infty L^*(i!\,\mathrm{ch}_i)$.

We finally remark that the cohomology $H^*(\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}; \mathbb{F}_p)$ has been calculated in [11] for all primes p. The result is quite complicated.

1.2. *A geometric formulation.* Let us first consider smooth *proper* maps $q : M^{d+n} \to X^n$ of smooth manifolds without boundary, for fixed $d \geq 0$, equipped with an orientation of $TM - q^*TX$, the (stable) relative tangent bundle. Two such maps $q_0\colon M_0 \to X$ and $q_1\colon M_1 \to X$ are *concordant* (traditionally, *cobordant*) if there exists a similar map $q_{\mathbb{R}}\colon W^{d+n+1} \to X \times \mathbb{R}$ transverse to $X \times \{0\}$ and $X \times \{1\}$, and such that the inverse images of $X \times \{0\}$ and $X \times \{1\}$ are isomorphic to q_0 and q_1 respectively, with all the relevant vector bundle data. The Pontryagin-Thom theory, cf. particularly [35], equates the set of concordance classes of such maps over fixed X with the set of homotopy classes of maps from X into the degree $-d$ term of the universal Thom spectrum,

$$\Omega^{\infty+d}\mathbf{M}\mathrm{SO} = \mathrm{colim}_n\ \Omega^{n+d}\mathrm{Th}\,(U_{n,\infty})\,.$$

The geometric reformulation of Theorem 1.1 is similar in spirit.

We consider smooth proper maps $q\colon M^{d+n} \to X^n$ much as before, together with a vector bundle epimorphism δq from $TM \times \mathbb{R}^i$ to $q^*TX \times \mathbb{R}^i$, where $i \gg 0$, and with an orientation of the d-dimensional kernel bundle of δq. (Note that δq is not required to agree with dq, the differential of q.) Again, the Pontryagin-Thom theory equates the set of concordance classes of such pairs $(q, \delta q)$ over fixed X with the set of homotopy classes of maps

$$X \longrightarrow \Omega^\infty \mathbf{hV}\,,$$

with $\Omega^\infty\mathbf{hV}$ as in (1.2). For a pair $(q, \delta q)$ as above which is integrable, $\delta q = dq$, the map q is a proper submersion with target X and hence a bundle of smooth closed d-manifolds on X by Ehresmann's fibration lemma [4, 8.12]. Thus the set of concordance classes of such integrable pairs over a fixed X is in natural bijection with the set of homotopy classes of maps

$$X \longrightarrow \coprod B\mathrm{Diff}(F^d)$$

where the disjoint union runs over a set of representatives of the diffeomorphism classes of closed, smooth and oriented d-manifolds. Comparing these two classification results we obtain a map

$$\alpha\colon \coprod B\mathrm{Diff}(F^d) \longrightarrow \mathbf{hV}$$

which for $d = 2$ is closely related to the map α_∞ of Theorem 1.1. The map α is not a homotopy equivalence (which is why we replace it by α_∞ when $d = 2$). However, using submersion theory we can refine our geometric understanding of homotopy classes of maps to $\mathbf{hV}$ and our understanding of α.

We suppose for simplicity that X is closed. As explained above, a homotopy class of maps from X to $\mathbf{hV}$ can be represented by a pair $(q, \delta q)$ with a proper $q\colon M \to X$, a vector bundle epimorphism $\delta q\colon TM \times \mathbb{R}^i \to q^*TX \times \mathbb{R}^i$ and an orientation on $\ker(\delta q)$. We set

$$E = M \times \mathbb{R}$$

and let $\bar{q}\colon E \to X$ be given by $\bar{q}(x,t) = q(x)$. The epimorphism δq determines an epimorphism $\delta\bar{q}\colon TE \times \mathbb{R}^i \to \bar{q}^*TX \times \mathbb{R}^i$. In fact, obstruction theory shows that we can take $i = 0$, and so we write $\delta\bar{q}\colon TE \to \bar{q}^*TX$. Since E is an open manifold, the submersion theorem of Phillips [34], [16], [15] applies, showing that the pair $(\bar{q}, \delta\bar{q})$ is homotopic through vector bundle surjections to a pair $(\pi, d\pi)$ consisting of a submersion $\pi\colon E \to X$ and $d\pi\colon TE \to \pi^*TX$. Let $f\colon E \to \mathbb{R}$ be the projection. This is proper; hence $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper.

The vertical tangent bundle $T^\pi E = \ker(d\pi)$ of π is identified with $\ker(\delta p) \cong \ker(\delta q) \times T\mathbb{R}$, so has a trivial line bundle factor. Let δf be the projection to that factor. In terms of the vertical or fiberwise 1-jet bundle,

$$p^1_\pi\colon J^1_\pi(E, \mathbb{R}) \longrightarrow E$$

whose fiber at $z \in E$ consists of all affine maps from the vertical tangent space $(T^\pi E)_z$ to $\mathbb{R}$, the pair $(f, \delta f)$ amounts to a section $\hat{f}$ of p^1_π such that $\hat{f}(z)\colon (T^\pi E)_z \to \mathbb{R}$ is surjective for every $z \in E$.

We introduce the notation $h\mathcal{V}(X)$ for the set of pairs $(\pi, \hat{f})$, where π is a smooth submersion $E \to X$ with $(d+1)$-dimensional oriented fibers and $\hat{f}\colon E \to J^1_\pi(E, \mathbb{R})$ is a section of p^1_π with underlying map $f\colon E \to \mathbb{R}$, subject to two conditions: for each $z \in E$ the affine map $\hat{f}(z)\colon (T^\pi E)_z \to \mathbb{R}$ is surjective, and $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper. Note that E is not fixed here.

Concordance defines an equivalence relation on $h\mathcal{V}(X)$. Let $h\mathcal{V}[X]$ be the set of equivalence classes. The arguments above lead to a natural bijection

$$h\mathcal{V}[X] \;\cong\; [X, \Omega^\infty \mathbf{hV}]\,. \tag{1.4}$$

We similarly define $\mathcal{V}(X)$ as the set of pairs (π, f) where $\pi\colon E \to X$ is a smooth submersion as before and $f\colon E \to \mathbb{R}$ is a smooth function, subject to two conditions: the restriction of f to any fiber of π is regular (= nonsingular), and $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper. Let $\mathcal{V}[X]$ be the correponding set of concordance classes. Since elements of $\mathcal{V}(X)$ are bundles of closed oriented d-manifolds over $X \times \mathbb{R}$, we have a natural bijection

$$\mathcal{V}[X] \;\cong\; [X, \coprod B\mathrm{Diff}(F^d)].$$

On the other hand an element $(\pi, f) \in \mathcal{V}(X)$ with $\pi\colon E \to X$ determines a section $j^1_\pi f$ of the projection $J^1_\pi E \to E$ by fiberwise 1-jet prolongation. The map

$$\mathcal{V}(X) \;\longrightarrow\; h\mathcal{V}(X)\,; \quad (\pi, f) \;\mapsto\; (\pi, j^1_\pi f) \tag{1.5}$$

respects the concordance relation and so induces a map $\mathcal{V}[X] \to h\mathcal{V}[X]$, which corresponds to α in (1.2).

1.3. *Outline of proof.* The main tool is a special case of the celebrated "first main theorem" of V.A. Vassiliev [45], [46] which can be used to approximate (1.5). We fix $d \geq 0$ as above. For smooth X without boundary we enlarge the set $\mathcal{V}(X)$ to the set $\mathcal{W}(X)$ consisting of pairs (π, f) with π as before but with $f\colon E \to \mathbb{R}$ a fiberwise Morse function rather than a fiberwise regular function. We keep the condition that the combined map $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper. There is a similar enlargement of $h\mathcal{V}(X)$ to a set $h\mathcal{W}(X)$. An element of $h\mathcal{W}(X)$ is a pair $(\pi, \hat{f})$ where $\hat{f}$ is a section of "Morse type" of the fiberwise 2-jet bundle $J^2_\pi E \to E$ with an underlying map f such that $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper. In analogy with (1.5), we have the 2-jet prolongation map

$$\mathcal{W}(X) \longrightarrow h\mathcal{W}(X)\,; \quad (\pi, f) \mapsto (\pi, j^2_\pi f)\,. \tag{1.6}$$

Dividing out by the concordance relation we get representable functors:

$$\mathcal{W}[X] \cong [X, |\mathcal{W}|\,]\,, \qquad h\mathcal{W}[X] \cong [X, |h\mathcal{W}|\,] \tag{1.7}$$

and (1.6) induces a map $j^2_\pi\colon |\mathcal{W}| \to |h\mathcal{W}|$. Vassiliev's first main theorem is a main ingredient in our proof (in Section 4) of

Theorem 1.2. *The jet prolongation map $|\mathcal{W}| \to |h\mathcal{W}|$ is a homotopy equivalence.*

There is a commutative square

$$\begin{array}{ccc} |\mathcal{V}| & \longrightarrow & |\mathcal{W}| \\ \downarrow & & \downarrow \\ |h\mathcal{V}| & \longrightarrow & |h\mathcal{W}|\,. \end{array} \tag{1.8}$$

We need information about the horizontal maps. This involves introducing "local" variants $\mathcal{W}_{\mathrm{loc}}(X)$ and $h\mathcal{W}_{\mathrm{loc}}(X)$ where we focus on the behavior of the functions f and jet bundle sections $\hat{f}$ near the fiberwise singularity set:

$$\begin{aligned} \Sigma(\pi, f) &= \{z \in E \mid df_z = 0 \text{ on } (T^\pi E)_z\}\,, \\ \Sigma(\pi, \hat{f}) &= \{z \in E \mid \text{ linear part of } \hat{f}(z) \text{ vanishes}\}. \end{aligned}$$

The localization is easiest to achieve as follows. Elements of $\mathcal{W}_{\mathrm{loc}}(X)$ are defined like elements (π, f) of $\mathcal{W}(X)$, but we relax the condition that $(\pi, f)\colon E \to X \times \mathbb{R}$ be proper to the condition that its restriction to $\Sigma(\pi, f)$ be proper. The definition of $h\mathcal{W}_{\mathrm{loc}}(X)$ is similar, and we obtain spaces $|\mathcal{W}_{\mathrm{loc}}|$ and $|h\mathcal{W}_{\mathrm{loc}}|$ which represent the corresponding concordance classes, together with a commutative diagram

$$\begin{array}{ccccc} |\mathcal{V}| & \longrightarrow & |\mathcal{W}| & \longrightarrow & |\mathcal{W}_{\mathrm{loc}}| \\ {\scriptstyle j^2_\pi}\downarrow & & {\scriptstyle j^2_\pi}\downarrow & & {\scriptstyle j^2_\pi}\downarrow \\ |h\mathcal{V}| & \longrightarrow & |h\mathcal{W}| & \longrightarrow & |h\mathcal{W}_{\mathrm{loc}}|. \end{array} \tag{1.9}$$

The next two theorems are proved in Section 3. They are much easier than Theorem 1.2.

THEOREM 1.3. *The jet prolongation map* $|\mathcal{W}_{\mathrm{loc}}| \to | h\mathcal{W}_{\mathrm{loc}}|$ *is a homotopy equivalence.*

THEOREM 1.4. *The maps* $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$ *define a homotopy fibration sequence of infinite loop spaces.*

The spaces $|h\mathcal{W}|$ and $|h\mathcal{W}_{\mathrm{loc}}|$ are, like $|h\mathcal{V}| = \Omega^\infty \mathbf{hV}$, colimits of certain iterated loop spaces of Thom spaces. Their homology can be approached by standard methods from algebraic topology.

The three theorems above are valid for any choice of $d \geq 0$. This is not the case for the final result that goes into the proof of Theorem 1.1, although many of the arguments leading to it are valid in general.

THEOREM 1.5. *For* $d = 2$, *the homotopy fiber of* $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ *is the space* $\mathbb{Z} \times B\Gamma_\infty^+$.

In conjunction with the previous three theorems this proves Theorem 1.1:

$$\mathbb{Z} \times B\Gamma_\infty^+ \;\simeq\; |h\mathcal{V}| \;\simeq\; \Omega^\infty \mathbf{hV} \;\simeq\; \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}\,.$$

The proof of Theorem 1.5 is technically the most demanding part of the paper. It rests on compatible stratifications of $|\mathcal{W}|$ and $|h\mathcal{W}|$, or more precisely on homotopy colimit decompositions

$$|\mathcal{W}| \;\simeq\; \mathrm{hocolim}_R\, |\mathcal{W}_R|\,, \quad |\mathcal{W}_{\mathrm{loc}}| \;\simeq\; \mathrm{hocolim}_R\, |\mathcal{W}_{\mathrm{loc},R}| \tag{1.10}$$

where R runs through the objects of a certain category of finite sets. The spaces $|\mathcal{W}_R|$ and $|\mathcal{W}_{\mathrm{loc},R}|$ classify certain bundle theories $\mathcal{W}_R(X)$ and $\mathcal{W}_{\mathrm{loc},R}(X)$. The proof of (1.10) is given in Section 5, and is valid for all $d \geq 0$. (Elements of $\mathcal{W}_R(X)$ are smooth fiber bundles $M^{n+d} \to X^n$ equipped with extra fiberwise "surgery data". The maps $\mathcal{W}_S(X) \to \mathcal{W}_R(X)$ induced contravariantly by morphisms $R \to S$ in the indexing category involve fiberwise surgeries on some of these data.)

The homotopy fiber of $|\mathcal{W}_R| \to |\mathcal{W}_{\mathrm{loc},R}|$ is a classifying space for smooth fiber bundles $M^{n+d} \to X^n$ with d-dimensional oriented fibers F^d, each fiber having its boundary identified with a disjoint union

$$\coprod_{r \in R} S^{\mu_r} \times S^{d-\mu_r-1}$$

where μ_r depends on $r \in R$. The fibers F^d need not be connected, but in Section 6 we introduce a modification $\mathcal{W}_{c,R}(X)$ of $\mathcal{W}_R(X)$ to enforce this additional property, keeping (1.10) almost intact. Again this works for all $d \geq 0$.

When $d = 2$ the homotopy fiber of $|\mathcal{W}_{c,R}| \to |\mathcal{W}_{\mathrm{loc},R}|$ becomes homotopy equivalent to $\coprod_g B\Gamma_{g,2|R|}$. A second modification of (1.10) which we undertake in Section 7 allows us to replace this by $\mathbb{Z} \times B\Gamma_{\infty,2|R|+1}$, functorially in R. It follows directly from Harer's theorem that these homotopy fibers are "independent" of R up to homology equivalences. Using an argument from [25] and [44] we conclude that the inclusion of any of these homotopy fibers $\mathbb{Z} \times B\Gamma_{\infty,2|R|+1}$ into the homotopy fiber of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ is a homology equivalence. This proves Theorem 1.5.

The paper is set up in such a way that it proves analogues of Theorem 1.1 for other classes of surfaces, provided that Harer type stability results have been established. This includes for example spin surfaces by the stability theorem of [1]. See also [10].

2. Families, sheaves and their representing spaces

2.1. *Language.* We will be interested in families of smooth manifolds, parametrized by other smooth manifolds. In order to formalize pullback constructions and gluing properties for such families, we need the language of sheaves. Let $\mathscr{X}$ be the category of smooth manifolds (without boundary, with a countable base) and smooth maps.

Definition 2.1. A *sheaf* on $\mathscr{X}$ is a contravariant functor $\mathcal{F}$ from $\mathscr{X}$ to the category of sets with the following property. For every open covering $\{U_i|i \in \Lambda\}$ of some X in $\mathscr{X}$, and every collection $(s_i \in F(U_i))_i$ satisfying $s_i|U_i \cap U_j = s_j|U_i \cap U_j$ for all $i, j \in \Lambda$, there is a unique $s \in F(X)$ such that $s|U_i = s_i$ for all $i \in \Lambda$.

In Definition 2.1, we do not insist that all of the U_i be nonempty. Consequently $\mathcal{F}(\emptyset)$ must be a singleton. For a disjoint union $X = X_1 \sqcup X_2$, the restrictions give a bijection $\mathcal{F}(X) \cong \mathcal{F}(X_1) \times \mathcal{F}(X_2)$. Consequently $\mathcal{F}$ is determined up to unique natural bijections by its behavior on connected nonempty objects X of $\mathscr{X}$.

For the sheaves $\mathcal{F}$ that we will be considering, an element of $\mathcal{F}(X)$ is typically a family of manifolds parametrized by X and with some additional structure. In this situation there is usually a sensible concept of *isomorphism* between elements of $\mathcal{F}(X)$, so that there might be a temptation to regard $\mathcal{F}(X)$ as a groupoid. We do not include these isomorphisms in our definition of $\mathcal{F}(X)$, however, and we do not suggest that elements of X should be confused with the corresponding isomorphism classes (since this would destroy the sheaf property). This paper is not about "stacks". All the same, we must ensure that our pullback and gluing constructions are well defined (and not just up to some sensible notion of isomorphism which we would rather avoid). This forces us to introduce the following purely set-theoretic concept. We fix, once and for all, a set Z whose cardinality is at least that of $\mathbb{R}$.

Definition 2.2. A map of sets $S \to T$ is *graphic* if it is a restriction of the projection $Z \times T \to T$. In particular, each graphic map with target T is determined by its source, which is a subset S of $Z \times T$.

Clearly, a graphic map f with target T is equivalent to a map from T to the power set $P(Z)$ of Z, which we may call the adjoint of f. Pullbacks of graphic maps are now easy to define: If $g\colon T_1 \to T_2$ is any map and $f\colon S \to T_2$ is a graphic map with adjoint $f^a\colon T \to P(Z)$, then the pullback $g^*f\colon g^*S \to T_1$ is, by definition, the graphic map with adjoint equal to the composition

$$T_1 \xrightarrow{g} T_2 \xrightarrow{f^a} P(Z). \tag{2.1}$$

If g is an identity, then $g^*S = S$ and $g^*f = f$; if g is a composition, $g = g_2 g_1$, then $g^*S = {g_1}^*{g_2}^*S$ and $g^*f = {g_1}^*{g_2}^*f$. Thus, with the above definitions, *base change* is associative.

Definition 2.3. Let $\mathrm{pr}\colon X\times\mathbb{R} \to X$ be the projection. Two elements s_0, s_1 of $\mathcal{F}(X)$ are *concordant* if there exist $s \in \mathcal{F}(X\times\mathbb{R})$ which agrees with pr^*s_0 on an open neighborhood of $X\times\,]-\infty,0]$ in $X\times\mathbb{R}$, and with pr^*s_1 on an open neighborhood of $X\times[1,+\infty[$ in $X\times\mathbb{R}$. The element s is then called a *concordance* from s_0 to s_1.

It is not hard to show that "being concordant" is an equivalence relation on the set $\mathcal{F}(X)$, for every X. We denote the set of equivalence classes by $\mathcal{F}[X]$. Then $X \mapsto \mathcal{F}[X]$ is still a contravariant functor on $\mathscr{X}$. It is practically never a sheaf, but it is *representable* in the following weak sense. There exists a space, denoted by $|\mathcal{F}|$, such that homotopy classes of maps from a smooth X to $|\mathcal{F}|$ are in natural bijection with the elements of $\mathcal{F}[X]$. This follows from very general principles expressed in Brown's representation theorem [3]. An explicit and more functorial construction of $|\mathcal{F}|$ will be described later. To us, $|\mathcal{F}|$ is more important than $\mathcal{F}$ itself. We define $\mathcal{F}$ in order to pin down $|\mathcal{F}|$.

Elements in $\mathcal{F}(X)$ can usually be regarded as *families* of elements in $\mathcal{F}(\star)$, parametrized by the manifold X. The space $|\mathcal{F}|$ should be thought of as a space which classifies families of elements in $\mathcal{F}(\star)$.

2.2. *Families with analytic data.* Let E be a smooth manifold, without boundary for now, and $\pi\colon E \to X$ a smooth map to an object of $\mathscr{X}$. The map π is a *submersion* if its differentials $TE_z \to TX_{\pi(z)}$ for $z \in E$ are all surjective. In that case, by the implicit function theorem, each fiber $E_x = \pi^{-1}(x)$ for $x \in X$ is a smooth submanifold of E, of codimension equal to $\dim(X)$. We remark that a submersion need not be surjective and a surjective submersion need not be a bundle. However, a *proper* smooth map $\pi\colon E \to X$ which is a submersion is automatically a smooth fiber bundle by Ehresmann's fibration lemma [4, Thm. 8.12].

In this paper, when we informally mention a *family* of smooth manifolds parametrized by some X in $\mathscr{X}$, we typically mean a submersion $\pi\colon E \to X$. The members of the family are then the fibers E_x of π. The *vertical tangent bundle* of such a family is the vector bundle $T^\pi E \to E$ whose fiber at $z \in E$ is the kernel of the differential $d\pi\colon TE_z \to TX_{\pi(z)}$.

To have a fairly general notion of orientation as well, we fix a space Θ with a right action of the infinite general linear group over the real numbers: $\Theta \times \mathrm{GL} \to \Theta$. For an n-dimensional vector bundle $W \to B$ let $\mathrm{Fr}(W)$ be the frame bundle, which we regard as a principal $\mathrm{GL}(n)$-bundle on B with $\mathrm{GL}(n)$ acting on the right.

Definition 2.4. By a *Θ-orientation* of W we mean a section of the associated bundle $(\mathrm{Fr}(W) \times \Theta)/\mathrm{GL}(n) \longrightarrow B$.

This includes a definition of a Θ-orientation on a finite dimensional real vector *space*, because a vector space is a vector bundle over a point.

Example 2.5. If Θ is a single point, then every vector bundle has a unique Θ-orientation. If Θ is $\pi_0(\mathrm{GL})$ with the action of GL by translation, then a Θ-orientation of a vector bundle is simply an orientation. (This choice of Θ is the one that will be needed in the proof of the Mumford conjecture.) If Θ is $\pi_0(\mathrm{GL}) \times Y$ for a fixed space Y, where GL acts by translation on $\pi_0(\mathrm{GL})$ and trivially on the factor Y, then a Θ-orientation on a vector bundle $W \to B$ is an orientation on W together with a map $B \to Y$.

Let $\widetilde{\mathrm{SL}}(n)$ be the universal cover of the special linear group $\mathrm{SL}(n)$. If $\Theta = \mathrm{colim}_n \Theta_n$ where Θ_n is the pullback of

$$E\mathrm{GL}(n) \longrightarrow B\mathrm{GL}(n) \longleftarrow B\widetilde{\mathrm{SL}}(n)\,,$$

then a Θ-orientation on a vector bundle W amounts to a spin structure on W. Here $E\mathrm{GL}(n)$ can be taken as the frame bundle associated with the universal n-dimensional vector bundle on $B\mathrm{GL}(n)$.

We also fix an integer $d \geq 0$. (For the proof of the Mumford conjecture, $d = 2$ is the right choice.) The data Θ and d will remain with us, fixed but unspecified, throughout the paper, except for Section 7 where we specialize to $d = 2$ and $\Theta = \pi_0\mathrm{GL}$.

Definition 2.6. For X in $\mathscr{X}$, let $\mathcal{V}(X)$ be the set of pairs (π, f) where $\pi\colon E \to X$ is a graphic submersion of fiber dimension $d+1$, with a Θ-orientation of its vertical tangent bundle, and $f\colon E \to \mathbb{R}$ is a smooth map, subject to the following conditions.

(i) The map $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper.

(ii) The map f is fiberwise nonsingular, i.e., the restriction of f to any fiber E_x of π is a nonsingular map.

For $(\pi, f) \in \mathcal{V}(X)$ with $\pi\colon E \to X$, the map $z \mapsto (\pi(z), f(z))$ from E to $X \times \mathbb{R}$ is a proper submersion and therefore a smooth bundle with d-dimensional fibers. The Θ-orientation on the vertical tangent bundle of π is equivalent to a Θ-orientation on the vertical tangent bundle of $(\pi, f)\colon E \to X \times \mathbb{R}$, since $T^\pi E \cong T^{(\pi,f)}E \times \mathbb{R}$. Consequently 2.6 is another way of saying that an element of $\mathcal{V}(X)$ is a bundle of smooth closed d-manifolds on $X \times \mathbb{R}$ with a Θ-orientation of its vertical tangent bundle. We prefer the formulation given in Definition 2.6 because it is easier to vary and generalize, as illustrated by our next definition.

Definition 2.7. For X in $\mathscr{X}$, let $\mathcal{W}(X)$ be the set of pairs (π, f) as in Definition 2.6, subject to condition (i) as before, but with condition (ii) replaced by the weaker condition

(iia) the map f is fiberwise Morse.

Recall that a smooth function $N \to \mathbb{R}$ is a Morse function precisely if its differential, viewed as a smooth section of the cotangent bundle $TN^* \to N$, is transverse to the zero section [12, II§6]. This observation extends to families. In other words, if $\pi\colon E \to X$ is a smooth submersion and $f\colon E \to \mathbb{R}$ is any smooth map, then f is fiberwise Morse if and only if the fiberwise differential of f, a section of the vertical cotangent bundle $T^\pi E^*$ on E, is fiberwise (over X) transverse to the zero section. This has the following consequence for the fiberwise singularity set $\Sigma(\pi, f) \subset E$ of f.

Lemma 2.8. *Suppose that $f\colon E \to \mathbb{R}$ is fiberwise Morse. Then $\Sigma(\pi, f)$ is a smooth submanifold of E and the restriction of π to $\Sigma(\pi, f)$ is a local diffeomorphism, alias* étale *map, from $\Sigma(\pi, f)$ to X.*

Proof. The fiberwise differential viewed as a section of the vertical cotangent bundle is transverse to the zero section. In particular $\Sigma = \Sigma(\pi, f)$ is a submanifold of E, of the same dimension as X. But moreover, the fiberwise Morse condition implies that for each $z \in \Sigma$, the tangent space $T\Sigma_z$ has trivial intersection in TE_z with the vertical tangent space $T^\pi E_z$. This means that Σ is transverse to each fiber of π, and also that the differential of $\pi|\Sigma$ at any point z of Σ is an invertible linear map $T\Sigma_z \to TX_{\pi(z)}$, and consequently that $\pi|\Sigma$ is a local diffeomorphism. □

Definition 2.9. For X in $\mathscr{X}$ let $\mathcal{W}_{\mathrm{loc}}(X)$ be the set of pairs (π, f), as in Definition 2.6, but replacing conditions (i) and (ii) by

(ia) the map $\Sigma(\pi, f) \to X \times \mathbb{R}$ defined by $z \mapsto (\pi(z), f(z))$ is *proper*,

(iia) f is *fiberwise Morse.*

2.3. *Families with formal-analytic data.* Let E be a smooth manifold and $p^k\colon J^k(E, \mathbb{R}) \to E$ the k-jet bundle, where $k \geq 0$. Its fiber $J^k(E, \mathbb{R})_z$ at $z \in E$ consists of equivalence classes of smooth map germs $f\colon (E, z) \to \mathbb{R}$, with f equivalent to g if the k-th Taylor expansions of f and g agree at z (in any local coordinates near z). The elements of $J^k(E,\mathbb{R})$ are called *k-jets* of maps from E to $\mathbb{R}$. The k-jet bundle $p^k\colon J^k(E,\mathbb{R}) \to E$ is a vector bundle.

Let $u\colon TE_z \to E$ be any exponential map at z, that is, a smooth map such that $u(0) = z$ and the differential at 0 is the identity $TE_z \to TE_z$. Then every jet $t \in J^k(E,\mathbb{R})_z$ can be represented by a unique germ $(E, z) \to \mathbb{R}$ whose composition with u is the germ at 0 of a polynomial function t_u of degree $\leq k$ on the vector space TE_z. The constant part (a real number) and the linear part (a linear map $TE_z \to \mathbb{R}$) of t_u do not depend on u. We call them the constant and linear part of t, respectively. If the linear part of t vanishes, then the quadratic part of t_u, which is a quadratic map $TE_z \to \mathbb{R}$, is again independent of u. We then call it the quadratic part of t.

Definition 2.10. A jet $t \in J^k(E,\mathbb{R})$ is *nonsingular* (assuming $k \geq 1$) if its linear part is nonzero. The jet t is *Morse* (assuming $k \geq 2$) if it has a nonzero linear part or, failing that, a nondegenerate quadratic part.

A smooth function $f\colon E \to \mathbb{R}$ induces a smooth section $j^k f$ of p^k, which we call the *k-jet prolongation* of f, following e.g. Hirsch [19]. (Some writers choose to call it the *k-jet* of f, which can be confusing.) Not every smooth section of p^k has this form. Sections of the form $j^k f$ are called *integrable*. Thus a smooth section of p^k is integrable if and only if it agrees with the k-jet prolongation of its underlying smooth map $f\colon E \to \mathbb{R}$.

We need a fiberwise version $J^k_\pi(E,\mathbb{R})$ of $J^k(E,\mathbb{R})$, fiberwise with respect to a submersion $\pi\colon E^{j+r} \to X^j$ with fibers E_x for $x \in X$. In a neighborhood of any $z \in E$ we may choose local coordinates $\mathbb{R}^j \times \mathbb{R}^r$ so that π becomes the projection onto $\mathbb{R}^j$ and $z = (0,0)$. Two smooth map germs $f, g\colon (E, z) \to \mathbb{R}$ define the same element of $J^k_\pi(E,\mathbb{R})_z$ if their k-th Taylor expansions in the $\mathbb{R}^r$ coordinates agree at $(0,0)$. Thus $J^k_\pi(E,\mathbb{R})_z$ is a quotient of $J^k(E,\mathbb{R})_z$ and $J^k_\pi(E,\mathbb{R})_z$ is identified with $J^k(E_{\pi(z)},\mathbb{R})$. There is a short exact sequence of vector bundles on E,

$$\pi^* J^k(X,\mathbb{R}) \longrightarrow J^k(E,\mathbb{R}) \longrightarrow J^k_\pi(E,\mathbb{R}).$$

Sections of the bundle projection $p^k_\pi\colon J^k_\pi(E,\mathbb{R}) \to E$ will be denoted $\hat{f}$, $\hat{g}$, ..., and their underlying functions from E to $\mathbb{R}$ by the corresponding letters f, g, and so on. Such a section $\hat{f}$ is *nonsingular*, resp. *Morse*, if $\hat{f}(z)$, viewed as an element of $J^k(E_{\pi(z)},\mathbb{R})$, is nonsingular, resp. Morse, for all $z \in E$.

Definition 2.11. The fiberwise singularity set $\Sigma(\pi, \hat{f})$ is the set of all $z \in E$ where $\hat{f}(z)$ is singular (assuming $k \geq 1$). Equivalently,

$$\Sigma(\pi, \hat{f}) = \hat{f}^{-1}(\Sigma_\pi(E,\mathbb{R}))\,,$$

where $\Sigma_\pi(E,\mathbb{R}) \subset J^2_\pi(E,\mathbb{R})$ is the submanifold consisting of the singular jets, i.e., those with vanishing linear part.

Again, any smooth function $f\colon E \to \mathbb{R}$ induces a smooth section $j^k_\pi f$ of p^k_π, which we call the *fiberwise k-jet prolongation* of f. The sections of the form $j^k_\pi f$ are called *integrable*. If $k \geq 1$ and $\hat{f}$ is integrable with $\hat{f} = j^k_\pi f$, then

$$\Sigma(\pi, \hat{f}) = \Sigma(\pi, f).$$

Definition 2.12. For an object X in $\mathscr{X}$, let $h\mathcal{V}(X)$ be the set of pairs $(\pi, \hat{f})$ where $\pi\colon E \to X$ is a graphic submersion of fiber dimension $d+1$, with a Θ-orientation of its vertical tangent bundle, and $\hat{f}$ is a smooth section of $p^2_\pi\colon J^2_\pi(E, \mathbb{R}) \to E$, subject to the following conditions:

(i) $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper.

(ii) $\hat{f}$ is fiberwise nonsingular.

Definition 2.13. For X in $\mathscr{X}$ let $h\mathcal{W}(X)$ be the set of pairs $(\pi, \hat{f})$, as in Definition 2.12, which satisfy condition (i), but where condition (ii) is replaced by the weaker condition

(iia) $\hat{f}$ is fiberwise Morse.

Definition 2.14. For X in $\mathscr{X}$ let $h\mathcal{W}_{\mathrm{loc}}(X)$ be the set of pairs $(\pi, \hat{f})$, as in Definition 2.12, but with conditions (i) and (ii) replaced by the weaker conditions

(ia) the map $\Sigma(\pi, \hat{f}) \to X \times \mathbb{R}\,;\; z \mapsto (\pi(z), f(z))$ is proper,

(iia) $\hat{f}$ is fiberwise Morse.

The six sheaves which we have so far defined, together with the obvious inclusion and jet prolongation maps, constitute a commutative square

(2.2)
$$\begin{array}{ccccc}
\mathcal{V} & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}_{\mathrm{loc}} \\
{\scriptstyle j^2_\pi}\downarrow & & {\scriptstyle j^2_\pi}\downarrow & & {\scriptstyle j^2_\pi}\downarrow \\
h\mathcal{V} & \longrightarrow & h\mathcal{W} & \longrightarrow & h\mathcal{W}_{\mathrm{loc}}.
\end{array}$$

2.4. *Concordance theory of sheaves.* Let $\mathcal{F}$ be a sheaf on $\mathscr{X}$ and let X be an object of $\mathscr{X}$. In 2.3, we defined the concordance relation on $\mathcal{F}(X)$ and introduced the quotient set $\mathcal{F}[X]$. It is necessary to have a relative version of $\mathcal{F}[X]$. Suppose that $A \subset X$ is a closed subset, where X is in $\mathscr{X}$. Let $s \in \mathrm{colim}_U \mathcal{F}(U)$ where U ranges over the open neighborhoods of A in $\mathscr{X}$. Note for example that any $z \in \mathcal{F}(\star)$ gives rise to such an element, namely $s = \{p^*_U(z)\}$ where $p_U\colon U \to \star$. In this case we often write z instead of s.

Definition 2.15. Let $\mathcal{F}(X,A;s) \subset \mathcal{F}(X)$ consist of the elements t in $\mathcal{F}(X)$ whose germ near A is equal to s. Two such elements t_0 and t_1 are concordant *relative to* A if they are concordant by a concordance whose germ near A is the constant concordance from s to s. The set of equivalence classes is denoted $\mathcal{F}[X,A;s]$.

We now construct the representing space $|\mathcal{F}|$ of $\mathcal{F}$ and list its most important properties. Let Δ be the category whose objects are the ordered sets $\underline{n} := \{0,1,2,\dots,n\}$ for $n \geq 0$, with order-preserving maps as morphisms. For $n \geq 0$ let $\Delta^n_e \subset \mathbb{R}^{n+1}$ be the extended standard n-simplex,

$$\Delta^n_e := \{(x_0,x_1,\dots,x_n) \in \mathbb{R}^{n+1} \mid \Sigma x_i = 1\}.$$

An order-preserving map $\underline{m} \to \underline{n}$ induces a map of affine spaces $\Delta^m_e \to \Delta^n_e$. This makes $\underline{n} \mapsto \Delta^n_e$ into a covariant functor from Δ to $\mathscr{X}$.

Definition 2.16. The *representing space* $|\mathcal{F}|$ of a sheaf $\mathcal{F}$ on $\mathscr{X}$ is the geometric realization of the simplicial set $\underline{n} \mapsto \mathcal{F}(\Delta^n_e)$.

An element $z \in \mathcal{F}(\star)$ gives a point $z \in |\mathcal{F}|$ and $\mathcal{F}[\star] = \pi_0|\mathcal{F}|$. In appendix A we prove that $|\mathcal{F}|$ represents the contravariant functor $X \mapsto \mathcal{F}[X]$. Indeed we prove the following slightly more general

PROPOSITION 2.17. *For X in $\mathscr{X}$, let $A \subset X$ be a closed subset and let $z \in \mathcal{F}(\star)$. There is a natural bijection ϑ from the set of homotopy classes of maps $(X,A) \to (\,|\mathcal{F}|, z)$ to the set $\mathcal{F}[X,A;z]$.*

Taking $X = S^n$ and A equal to the base point, we see that the homotopy group $\pi_n(|\mathcal{F}|, z)$ is identified with the set of concordance classes $\mathcal{F}[S^n, \star; z]$. We introduce the notation

$$\pi_n(\mathcal{F}, z) := \mathcal{F}[S^n, \star; z]\,.$$

A map $v\colon \mathcal{E} \to \mathcal{F}$ of sheaves induces a map $|v|\colon |\mathcal{E}| \to |\mathcal{F}|$ of representing spaces. We call v a *weak equivalence* if $|v|$ is a homotopy equivalence.

PROPOSITION 2.18. *Let $v\colon \mathcal{E} \to \mathcal{F}$ be a map of sheaves on $\mathscr{X}$. Suppose that v induces a surjective map*

$$\mathcal{E}[X,A;s] \longrightarrow \mathcal{F}[X,A;v(s)]$$

for every X in $\mathscr{X}$ with a closed subset $A \subset X$ and any germ $s \in \operatorname{colim}_U \mathcal{E}(U)$, where U ranges over the neighborhoods of A in X. Then v is a weak equivalence.

Proof. The hypothesis implies easily that the induced map $\pi_0\mathcal{E} \to \pi_0\mathcal{F}$ is onto and that, for any choice of base point $z \in \mathcal{E}(\star)$, the map of concordance sets $\pi_n(\mathcal{E}, z) \to \pi_n(\mathcal{F}, v(z))$ induced by v is bijective. Indeed, to see that v induces a surjection $\pi_n(\mathcal{E}, z) \to \pi_n(\mathcal{F}, v(z))$, simply take $(X, A, s) = (S^n, \star, z)$. To see that an element $[t]$ in the kernel of this surjection is zero, take $X = \mathbb{R}^{n+1}$, $A = \{z \in \mathbb{R}^{n+1} \mid \|z\| \geq 1\}$ and $s = p^*t$ where $p\colon \mathbb{R}^{n+1} \smallsetminus \{0\} \to S^n$ is the radial projection. The hypothesis that $[t]$ is in the kernel amounts to a null-concordance for $v(t)$ which can be reformulated as an element of $\mathcal{F}[X, A; v(s)]$. Our assumption on v gives us a lift of that element to $\mathcal{E}[X, A; s]$ which in turn can be interpreted as a null-concordance of t. □

Applying the representing space construction to the sheaves displayed in diagram (2.2), we get the commutative diagram (1.9) from the introduction.

2.5. *Some useful concordances.*

LEMMA 2.19 (Shrinking lemma). *Let (π, f) be an element of $\mathcal{V}(X)$, $\mathcal{W}(X)$ or $\mathcal{W}_{\mathrm{loc}}(X)$, with $\pi\colon E \to X$ and $f\colon E \to \mathbb{R}$. Let $e\colon X \times \mathbb{R} \to \mathbb{R}$ be a smooth map such that, for any $x \in X$, the map $e_x\colon \mathbb{R} \to \mathbb{R}$ defined by $t \mapsto e(x,t)$ is an orientation preserving embedding. Let $E^{(1)} = \{z \in E \mid f(z) \in e_{\pi(z)}(\mathbb{R})\}$. Let*

$$\pi^{(1)} = \pi | E^{(1)} \qquad \textit{and} \qquad f^{(1)}(z) = {e_{\pi(z)}}^{-1} f(z)$$

for $z \in E^{(1)}$. Then (π, f) is concordant to $(\pi^{(1)}, f^{(1)})$.

Proof. Choose an $\varepsilon > 0$ and a smooth family of smooth embeddings $u_{(x,t)}\colon \mathbb{R} \to \mathbb{R}$, where $t \in \mathbb{R}$ and $x \in X$, such that $u_{(x,t)} = \mathrm{id}$ whenever $t < \varepsilon$ and $u_{(x,1)} = e_x$ whenever $t > 1 - \varepsilon$. Let

$$E^{(\mathbb{R})} = \left\{(z,t) \in E \times \mathbb{R} \,\middle|\, f(z) \in u_{(\pi(z),t)}(\mathbb{R})\right\}.$$

Then $(z,t) \mapsto (\pi(z), t)$ defines a smooth submersion $\pi^{(\mathbb{R})}$ from $E^{(\mathbb{R})}$ to $X \times \mathbb{R}$, and

$$z \quad \mapsto \quad {u_{(\pi(z),t)}}^{-1} f(z)$$

defines a smooth map $f^{(\mathbb{R})}\colon E^{(\mathbb{R})} \to \mathbb{R}$. Now $(\pi^{(\mathbb{R})}, f^{(\mathbb{R})})$ is a concordance from (π, f) to $(\pi^{(1)}, f^{(1)})$, modulo some simple re-labelling of the elements of $E(\mathbb{R})$ to ensure that $\pi^{(\mathbb{R})}$ is graphic. (As it stands, E is a subset of $Z \times X$, compare 2.2, and $E^{(\mathbb{R})}$ is a subset of $(Z \times X) \times \mathbb{R}$. But we want $E^{(\mathbb{R})}$ to be a subset of $Z \times (X \times \mathbb{R})$; hence the need for relabelling.) □

Lemma 2.19 has an obvious analogue for the sheaves $h\mathcal{V}$, $h\mathcal{W}$ and $h\mathcal{W}_{\mathrm{loc}}$, which we do not state explicitly.

LEMMA 2.20. *Every class in $\mathcal{W}[X]$ or $h\mathcal{W}[X]$ has a representative (π, f), resp. $(\pi, \hat{f})$, in which $f\colon E \to \mathbb{R}$ is a bundle projection, so that*

$$E \cong f^{-1}(0) \times \mathbb{R}.$$

Proof. We concentrate on the first case, starting with an arbitrary (π, f) in $\mathcal{W}[X]$. We do not assume that $f: E \to \mathbb{R}$ is a bundle projection to begin with. However, by Sard's theorem we can find a regular value $c \in \mathbb{R}$ for f. The singularity set of f (not to be confused with the fiberwise singularity set of f) is closed in E. Therefore its image under the proper map $(\pi, f): E \to X \times \mathbb{R}$ is closed. (Proper maps between locally compact spaces are closed maps). The complement of that image is an open neighborhood U of $X \times \{c\}$ in $X \times \mathbb{R}$ containing no critical points of f. It follows easily that there exists $e: X \times \mathbb{R} \to \mathbb{R}$ as in Lemma 2.19, with $e(x, 0) = c$ for all x and $(x, e(x,t)) \in U$ for all $x \in X$ and $t \in \mathbb{R}$. Apply Lemma 2.19 with this choice of e. In the resulting $(\pi^{(1)}, f^{(1)}) \in \mathcal{W}(X)$, the map $f^{(1)}: E^{(1)} \to \mathbb{R}$ is nonsingular and proper, hence a bundle projection. (It is not claimed that $f^{(1)}$ is fiberwise nonsingular.) □

We now introduce two sheaves $\mathcal{W}^0$ and $h\mathcal{W}^0$ on $\mathscr{X}$. They are weakly equivalent to $\mathcal{W}$ and $h\mathcal{W}$, respectively, but better adapted to Vassiliev's integrability theorem, as we will explain in Section 4.

Definition 2.21. For X in $\mathscr{X}$ let $\mathcal{W}^0(X)$ be the set of all pairs (π, f) as in Definition 2.7, replacing however condition (iia) there by the weaker

(iib) f is fiberwise Morse in some neighborhood of $f^{-1}(0)$.

Definition 2.22. For X in $\mathscr{X}$ let $h\mathcal{W}^0(X)$ be the set of all pairs $(\pi, \hat{f})$ as in Definition 2.13, replacing however condition (iia) by the weaker

(iib) $\hat{f}$ is fiberwise Morse in some neighborhood of $f^{-1}(0)$.

From the definition, there are inclusions $\mathcal{W} \to \mathcal{W}^0$ and $h\mathcal{W} \to h\mathcal{W}^0$. There is also a jet prolongation map $\mathcal{W}^0 \to h\mathcal{W}^0$ which we may regard as an inclusion, the inclusion of the subsheaf of integrable elements.

LEMMA 2.23. *The inclusions $\mathcal{W} \to \mathcal{W}^0$ and $h\mathcal{W} \to h\mathcal{W}^0$ are weak equivalences.*

Proof. We will concentrate on the first of the two inclusions, $\mathcal{W} \to \mathcal{W}^0$. Fix (π, f) in $\mathcal{W}^0(X)$, with $\pi: E \to X$ and $f: E \to \mathbb{R}$. We will subject (π, f) to a concordance ending in $\mathcal{W}(X)$. Choose an open neighborhood U of $f^{-1}(0)$ in E such that, for each $x \in X$, the critical points of $f_x = f|E_x$ on $E_x \cap U$ are all nondegenerate. Since $E \smallsetminus U$ is closed in E and the map $(\pi, f): E \to X \times \mathbb{R}$ is proper, the image of $E \smallsetminus U$ under that map is a closed subset of $X \times \mathbb{R}$ which has empty intersection with $X \times 0$. Again it follows that a map $e: X \times \mathbb{R} \to \mathbb{R}$ as in 2.19 can be constructed such that $e(x, 0) = 0$ for all x and $(x, e(x,t)) \in U$ for all $(x,t) \in X \times \mathbb{R}$. As in the proof of Lemma 2.19, use e to construct a concordance from (π, f) to some element $(\pi^{(1)}, f^{(1)})$ which, by inspection, belongs to $\mathcal{W}(X)$. If the restriction of (π, f) to an open neighborhood Y_1 of a closed $A \subset X$ belongs to $\mathcal{W}(Y_1)$, then the concordance can be made relative to Y_0, where Y_0 is a smaller open neighborhood of A in X. □

3. The lower row of diagram (1.9)

This section describes the homotopy types of the spaces in the lower row of (1.9) in bordism-theoretic terms. One of the conclusions is that the lower row is a homotopy fiber sequence, proving Theorem 1.4. We also show that the jet prolongation map $|\mathcal{W}_{\mathrm{loc}}| \to | h\mathcal{W}_{\mathrm{loc}}|$ is a homotopy equivalence (the fact as such does not belong in this section, but its proof does). In the standard case where $d = 2$ and $\Theta = \pi_0(\mathrm{GL})$, the space $|h\mathcal{V}|$ will be identified with $\Omega^\infty \mathbb{C}P^\infty_{-1}$.

3.1. *A cofiber sequence of Thom spectra.* Let $\mathrm{G}\mathcal{W}(d+1, n)$ be the space of triples (V, ℓ, q) consisting of a Θ-oriented $(d+1)$-dimensional linear subspace $V \subset \mathbb{R}^{d+1+n}$, a linear map $\ell: V \to \mathbb{R}$ and a quadratic form $q: V \to \mathbb{R}$, subject to the condition that if $\ell = 0$, then q is nondegenerate. $\mathrm{G}\mathcal{W}(d+1, n)$ classifies $(d+1)$-dimensional Θ-oriented vector bundles whose fibers have the above extra structure; i.e., each fiber V comes equipped with a Morse type map $\ell + q: V \to \mathbb{R}$ and with a linear embedding into $\mathbb{R}^{d+1+n}$.

The tautological $(d+1)$-dimensional vector bundle U_n on $\mathrm{G}\mathcal{W}(d+1, n)$ is canonically embedded in a trivial bundle $\mathrm{G}\mathcal{W}(d+1, n) \times \mathbb{R}^{d+1+n}$. Let

$$U_n^\perp \quad \subset \quad \mathrm{G}\mathcal{W}(d+1, n) \times \mathbb{R}^{d+1+n}$$

be the orthogonal complement bundle, an n-dimensional vector bundle on $\mathrm{G}\mathcal{W}(d+1, n)$. The tautological bundle U_n comes equipped with the extra structure consisting of a map from (the total space of) U_n to $\mathbb{R}$ which, on each fiber of U_n, is a Morse type map. (The fiber of U_n over a point $(V, q, \ell) \in \mathrm{G}\mathcal{W}(d+1, n)$ is identified with the $(d+1)$-dimensional vector space V and the map can then be described as $\ell + q$.)

Let $S(\mathbb{R}^{d+1})$ be the vector space of quadratic forms on $\mathbb{R}^{d+1}$ (or equivalently, symmetric $(d+1) \times (d+1)$ matrices) and $\Delta \subset S(\mathbb{R}^{d+1})$ the subspace of the degenerate forms (not a linear subspace). The complement $Q(\mathbb{R}^{d+1}) = S(\mathbb{R}^{d+1}) \smallsetminus \Delta$ is the space of nondegenerate quadratic forms on $\mathbb{R}^{d+1}$. Since quadratic forms can be diagonalized,

$$Q(\mathbb{R}^{d+1}) = \coprod_{i=0}^{d+1} Q(i, d+1-i)$$

where $Q(i, d+1-i)$ is the connected component containing the form q_i given by

$$q_i(x_1, x_2, \ldots, x_{d+1}) = -(x_1^2 + \cdots + x_i^2) + (x_{i+1}^2 + \cdots + x_{d+1}^2).$$

The stabilizer $\mathrm{O}(i, d+1-i)$ of q_i for the (transitive) action of $\mathrm{GL}(d+1)$ on $Q(i, d+1-i)$ has $\mathrm{O}(i) \times \mathrm{O}(d+1-i)$ as a maximal compact subgroup and $\mathrm{GL}(d+1)$ has $\mathrm{O}(d+1)$ as a maximal compact subgroup. Hence the inclusion

$$(\mathrm{O}(i) \times \mathrm{O}(d+1-i))\backslash \mathrm{O}(d+1) \quad \longrightarrow \quad Q(i, d+1-i)\,; \quad \text{coset of } g \quad \mapsto \quad q_i g$$

is a homotopy equivalence, and therefore the subspace

$$\text{(3.1)} \qquad \begin{aligned} Q^0(\mathbb{R}^{d+1}) \quad &= \quad \{q_0, q_1, \ldots, q_{d+1}\} \cdot \mathrm{O}(d+1) \\ &\cong \quad \textstyle\coprod_{i=0}^{d+1} (\mathrm{O}(i) \times \mathrm{O}(d+1-i))\backslash \mathrm{O}(d+1) \end{aligned}$$

of $Q(\mathbb{R}^{d+1})$ is a deformation retract, $Q(\mathbb{R}^{d+1}) \simeq Q^0(\mathbb{R}^d)$.

For the submanifold $\Sigma(d+1, n) \subset \mathrm{G}\mathcal{W}(d+1, n)$ consisting of the triples (V, ℓ, q) with $\ell = 0$ we have

$$\text{(3.2)} \qquad \Sigma(d+1, n) \quad \cong \quad \big(\mathrm{O}(d+1+n)/\mathrm{O}(n) \times Q(\mathbb{R}^{d+1}) \times \Theta\big)/\mathrm{O}(d+1)\,.$$

The restriction of U_n to $\Sigma(d+1, n)$ comes equipped with the extra structure of a fiberwise nondegenerate quadratic form. There is a canonical normal bundle for $\Sigma(d+1, n)$ in $\mathrm{G}\mathcal{W}(d+1, n)$ which is easily identified with the dual bundle $U_n^*|\Sigma(d+1, n)$. Hence there is a homotopy cofiber sequence

$$\mathrm{G}\mathcal{V}(d+1, n) \hookrightarrow \mathrm{G}\mathcal{W}(d+1, n) \longrightarrow \mathrm{Th}\,(U_n^*|\Sigma(d+1, n))$$

where $\mathrm{G}\mathcal{V}(d+1, n) = \mathrm{G}\mathcal{W}(d+1, n) \smallsetminus \Sigma(d+1, n)$ and $\mathrm{Th}\,(\ldots)$ denotes the Thom space. This leads to a homotopy cofiber sequence of Thom spaces

$$\mathrm{Th}\,(U_n^\perp|\mathrm{G}\mathcal{V}(d+1, n)) \longrightarrow \mathrm{Th}\,(U_n^\perp) \longrightarrow \mathrm{Th}\,(U_n^\perp \oplus U_n^*|\Sigma(d+1, n)).$$

(A *homotopy cofiber sequence* is a diagram $A \to B \to C$ of spaces, where C is pointed, together with a nullhomotopy of the composite map $A \to C$ such that the resulting map from $\mathrm{cone}(A \to B)$ to C is a weak homotopy equivalence.)

We view the space $\mathrm{Th}\,(U_n^\perp)$ as the $(n+d)$-th space in a spectrum $\mathbf{hW}$, and similarly for the other two Thom spaces. Then as n varies the sequence above becomes a homotopy cofiber sequence of spectra

$$\mathbf{hV} \longrightarrow \mathbf{hW} \longrightarrow \mathbf{hW}_{\mathrm{loc}}.$$

We then have the corresponding infinite loop spaces

$$\begin{aligned} \Omega^\infty \mathbf{hV} \quad &= \quad \mathrm{colim}_n\, \Omega^{d+n} \mathrm{Th}\,(U_n^\perp|\mathrm{G}\mathcal{V}(d+1, n))\,, \\ \Omega^\infty \mathbf{hW} \quad &= \quad \mathrm{colim}_n\, \Omega^{d+n} \mathrm{Th}\,(U_n^\perp)\,, \\ \Omega^\infty \mathbf{hW}_{\mathrm{loc}} \quad &= \quad \mathrm{colim}_n\, \Omega^{d+n} \mathrm{Th}\,(U_n^\perp \oplus U_n^*|\Sigma(d+1, n)). \end{aligned}$$

(We use CW-models for the spaces involved. For example, $\Omega^{d+n}\mathrm{Th}\,(U_n^\perp)$ can be considered as the representing space of the sheaf on $\mathscr{X}$ which to a smooth X associates the set of pointed maps from $X_+ \wedge S^{d+n}$ to $\mathrm{Th}\,(U_n^\perp)$. The representing space is a CW-space.)

The homotopy cofiber sequence of spectra above yields a homotopy fiber sequence of infinite loop spaces

$$\Omega^\infty \mathbf{hV} \longrightarrow \Omega^\infty \mathbf{hW} \longrightarrow \Omega^\infty \mathbf{hW}_{\mathrm{loc}}\ , \tag{3.3}$$

that is, $\Omega^\infty \mathbf{hV}$ is homotopy equivalent to the homotopy fiber of the right-hand map. (A *homotopy fiber sequence* is a diagram of spaces $A \to B \to C$, where C is pointed, together with a nullhomotopy of the composite map $A \to C$ such that the resulting map from A to hofiber$(B \to C)$ is a weak homotopy equivalence.) In particular there is a long exact sequence of homotopy groups associated with diagram (3.3) and a Leray-Serre spectral sequence of homology groups.

Suppose that a topological group G acts on a space Q from the right. We use the notation Q_{hG} for the "Borel construction" or homotopy orbit space $Q \times_G EG$, where EG is a contractible space with a free G-action.

LEMMA 3.1. *There is a homotopy equivalence of infinite loop spaces*

$$\Omega^\infty \mathbf{hW}_{\mathrm{loc}} \simeq \Omega^\infty S^{1+\infty}(\Sigma(d+1,\infty)_+)$$

where $(\Sigma(d+1,\infty)$ *is a disjoint union of homotopy orbit spaces*,

$$\Sigma(d+1,\infty) \simeq \coprod_{i=0}^{d+1} \Theta_{h\mathrm{O}(i,d+1-i)}.$$

Proof. Since $U_n|\Sigma(d{+}1,n)$ comes equipped with a fiberwise nondegenerate quadratic form, $U_n^*|\Sigma(d+1,n)$ is canonically identified with $U_n|\Sigma(d+1,n)$. Consequently the restriction

$$U_n^\perp \oplus U_n^* \,\big|\, \Sigma(d+1,n)$$

is trivialized, so that $\mathrm{Th}\,(U_n^\perp \oplus U_n^* \,\big|\, \Sigma(d+1,n)) \;\simeq\; S^{d+1+n}(\Sigma(d+1,n)_+)\,.$ Hence

$$\Omega^\infty \mathbf{hW}_{\mathrm{loc}} \;\simeq\; \Omega^\infty S^{1+\infty}(\Sigma(d+1,\infty)_+)$$

where $\Sigma(d+1,\infty) = \bigcup \Sigma(d+1,n)$. Using the description (3.2) of $\Sigma(d+1,n)$ and the equivariant homotopy equivalence $Q(\mathbb{R}^{d+1}) \simeq Q^0(\mathbb{R}^{d+1})$, see (3.1), we get

$$\Sigma(d+1,n) \;\simeq\; \Big(\mathrm{O}(d+1+n)/\mathrm{O}(n)) \times Q^0(\mathbb{R}^{d+1}) \times \Theta\Big)/\mathrm{O}(d+1).$$

The union $\bigcup_n \mathrm{O}(d+1+n)/\mathrm{O}(n)$ is a contractible free $\mathrm{O}(d+1)$-space, so that $\Sigma(d+1,\infty)$ is homotopy equivalent to the homotopy orbit space of the canonical right action of $\mathrm{O}(d+1)$ on the space

$$Q^0(\mathbb{R}^{d+1}) \times \Theta \;\cong\; \left(\coprod_{i=0}^{d+1} (\mathrm{O}(i) \times \mathrm{O}(d+1-i))\backslash \mathrm{O}(d+1)\right) \times \Theta\,.$$

That in turn is homotopy equivalent to the disjoint union over i of the homotopy orbit spaces of $\mathrm{O}(i) \times \mathrm{O}(d+1-i) \simeq \mathrm{O}(i, d+1-i)$ acting on the left of $(O(d+1) \times \Theta)/O(d+1) \cong \Theta$. $\square$

Let $\mathrm{G}(d, n; \Theta)$ be the space of d-dimensional Θ-oriented linear subspaces in $\mathbb{R}^{d+n}$. It can be identified with a subspace of $\mathrm{G}\mathcal{V}(d+1, n)$, consisting of the $(V, \ell + q)$ where V contains the subspace $\mathbb{R} \times 0 \times 0$ of $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n$, and $\ell + q$ is the linear projection to that subspace (so that $q = 0$). The injection is covered by a fiberwise isomorphism of vector bundles

$$T_n^\perp \longrightarrow U_n^\perp \,\big|\, \mathrm{G}\mathcal{V}(d+1, n)$$

where $T_n^\perp$ is the standard n-plane bundle on $\mathrm{G}(d, n; \Theta)$.

Lemma 3.2. *The induced map of Thom spaces*

$$\mathrm{Th}\,(T_n^\perp) \longrightarrow \mathrm{Th}\,(U_n^\perp \,|\, \mathrm{G}\mathcal{V}(d+1, n))$$

is $(d + 2n - 1)$*-connected. Hence* $\Omega^\infty \mathbf{hV} \simeq \mathrm{colim}_n\, \Omega^{d+n} \mathrm{Th}\,(T_n^\perp)$.

Proof. It is enough to show that the inclusion of $\mathrm{G}(d, n; \Theta)$ in $\mathrm{G}\mathcal{V}(d+1, n)$ is $(d+n-1)$-connected. Viewing both of these spaces as total spaces of certain bundles with fiber Θ reduces the claim to the case where Θ is a single point. Note also that $\mathrm{G}\mathcal{V}(d+1, n)$ has a deformation retract consisting of the pairs $(V, \ell + q)$ with $q = 0$ and $\|\ell\| = 1$. This deformation retract is homeomorphic to the coset space $\mathrm{O}(d) \times \mathrm{O}(n) \backslash \mathrm{O}(1 + d + n)$, when we assume that $\Theta = \star$. We are therefore looking at the inclusion of $(\mathrm{O}(d) \times \mathrm{O}(n)) \backslash \mathrm{O}(d+n)$ in $(\mathrm{O}(d) \times \mathrm{O}(n)) \backslash \mathrm{O}(1 + d + n)$, which is indeed $(d + n - 1)$-connected. $\square$

In the standard case where $d = 2$ and $\Theta = \pi_0 \mathrm{GL}$, we may compare the Grassmannian of oriented planes $\mathrm{G}(2, 2n; \Theta)$ with the complex projective n-space. The map

$$\mathbb{C}P^n \longrightarrow G(2, 2n; \Theta)$$

that forgets the complex structure is $(2n - 1)$-connected. The pullback of $T_{2n}^\perp$ under this map is the realification of the tautological complex n-plane bundle $L_n^\perp$ and the associated map of Thom spaces is $(4n - 1)$-connected. The spectrum $\mathbb{C}P^\infty_{-1}$ with $(2n+2)$-nd space $\mathrm{Th}\,(L_n^\perp)$ is therefore weakly equivalent to the Thom spectrum $\mathbf{hV}$. We can now collect the main conclusions of this section, 3.1, in

Proposition 3.3. *For* $d = 2$ *and* $\Theta = \pi_0 \mathrm{GL}$, *the homotopy fiber sequence* (3.3) *is homotopy equivalent to*

$$\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} \longrightarrow \Omega^\infty \mathbf{hW} \longrightarrow \Omega^\infty S^{1+\infty}\big(\big(\coprod_{i=0}^{3} B\mathrm{SO}(i, 3-i)\big)_+\big),$$

where $\mathrm{SO}(i, 3 - i) = \mathrm{SO}(3) \cap \mathrm{O}(i, 3 - i)$. $\square$

3.2. *The spaces $|h\mathcal{W}|$ and $|h\mathcal{V}|$.* In Section 2.3 we described the jet bundle $J^2(E,\mathbb{R})$ and its fiberwise version as certain spaces of smooth map germs $(E,z)\to\mathbb{R}$, modulo equivalence. For our use in this section and the next it is better to view it as a construction on the tangent bundle. For a vector space V, let $J^2(V)$ denote the vector space of maps

$$\hat{f}\colon \quad V\to \quad \mathbb{R}\,, \qquad \hat{f}(v) \quad = \quad c+\ell(v)+q(v)$$

where $c\in\mathbb{R}$ is a constant, $\ell\in V^*$ and $q\colon V\to\mathbb{R}$ is a quadratic map. This is a contravariant continuous functor on vector spaces, so extends to a functor on vector bundles with $J^2(F)_z = J^2(F_z)$.

When $F = TE$ is the tangent bundle of a manifold E, then there is an isomorphism of vector bundles

$$J^2(E,\mathbb{R})\cong J^2(TE).$$

Indeed after a choice of a connection on TE, the associated exponential map induces a diffeomorphism germ $\exp_z\colon(TE_z,0)\to(E,z)$. Composition with $\exp_z$ is an isomorphism from $J^2(E,\mathbb{R})_z$ to $J^2(TE_z)$.

LEMMA 3.4. *Let $\pi\colon E\to X$ be a smooth submersion. Any choice of connection on the vertical tangent bundle $T^\pi E$ induces an isomorphism*

$$J^2_\pi(E,\mathbb{R})\longrightarrow J^2(T^\pi E).$$

This is natural under pullbacks of submersions.

Proof. In addition to choosing a connection on $T^\pi E$, we may choose a smooth linear section of the vector bundle surjection $d\pi : TE\to\pi^*TX$ and a connection on TX. This leads to a splitting

$$TE\cong T^\pi E\oplus\pi^*TX$$

and determines a direct sum connection on TE. The associated exponential diffeomorphism germ $\exp\colon(TE_z,0)\longrightarrow(E,z)$ is fiberwise, i.e., it restricts to a diffeomorphism germ

$$((T^\pi E)_z,0)\to(E_{\pi(z)},z) \tag{3.4}$$

for each $z\in E$. Indeed, the chosen connection on $T^\pi E$ restricts to a connection on the tangent bundle of $E_{\pi(z)}$, and any geodesic in $E_{\pi(z)}$ for that connection is clearly a geodesic in E as well. The argument also shows that the diffeomorphism germ (3.4), and the isomorphism $J^2_\pi(E,\mathbb{R})_z\longrightarrow J^2(T^\pi E)_z$ which it induces, depend only on the choice of a connection on $T^\pi E$, but not on the choice of a splitting of $d\pi : TE\to\pi^*TX$ and a connection on TX. (However, making use of all the choices, we arrive at a commutative diagram of vector bundles

$$\begin{tikzcd} J^2(TE) \arrow[r, two heads, "i^*"] \arrow[d, "\cong"] & J^2(T^\pi E) \arrow[d, "\cong"] \\ J^2(E,\mathbb{R}) \arrow[r, two heads, "i^*"] & J^2_\pi(E,\mathbb{R}) \end{tikzcd}$$

where the horizontal epimorphisms are induced by inclusions.) Finally, if

$$\begin{tikzcd} \varphi^* E \arrow[r, "\bar\varphi"] \arrow[d, "\varphi^*\pi"] & E \arrow[d, "\pi"] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

is a pullback diagram of submersions, then a choice of connection on $T^\pi E$ determines a connection on $\bar\varphi^* T^\pi E \cong T^{\varphi^*\pi}\varphi^* E$. The resulting exponential diffeomorphism germs are related by a commutative diagram

$$\begin{tikzcd} ((T^{\varphi^*\pi}\varphi^* E)_z, 0) \arrow[r] \arrow[d, "d^\pi\bar\varphi"] & (\varphi^* E_{\varphi^*\pi(z)}, z) \arrow[d, "\bar\varphi"] \\ ((T^\pi E)_{\bar\varphi(z)}, 0) \arrow[r] & (E_{\pi\bar\varphi(z)}, \bar\varphi(z)) . \end{tikzcd}$$

This proves the naturality claim. □

We can re-define $h\mathcal{W}(X)$ in Definition 2.13 as the set of certain pairs $(\pi, \hat f)$ much as before, with $\pi\colon E \to X$, where $\hat f$ is now a Morse type section of $J^2(T^\pi E)$. The above lemma tells us that the new definition of $h\mathcal{W}$ is related to the old one by a chain of two weak equivalences. (In the middle of that chain is yet another variant of $h\mathcal{W}(X)$, namely the set of triples $(\pi, \hat f, \nabla)$ where π and $\hat f$ are as in Definition 2.13, while ∇ is a connection on $T^\pi E$.)

Our object now is to construct a natural map

$$\tau\colon h\mathcal{W}[X] \longrightarrow [X, \Omega^\infty \mathbf{hW}]. \tag{3.5}$$

Here [,] in the right-hand side denotes a set of homotopy classes of maps.

We assume familiarity with the Pontryagin-Thom relationship between Thom spectra and their infinite loop spaces on the one hand, and bordism theory on the other. One direction of this relies on transversality theorems; the other uses collapse maps to normal bundles of submanifolds in euclidean spaces. See [43] and especially [35]. Applied to our situation this identifies $[X, \Omega^\infty \mathbf{hW}]$ with a group of bordism classes of certain triples $(M, g, \hat g)$. Here M is smooth without boundary, $\dim(M) = \dim(X) + d$, and $g, \hat g$ together constitute a vector bundle pullback square

$$\begin{tikzcd} TM \times \mathbb{R} \times \mathbb{R}^j \arrow[r, "\hat{g}"] \arrow[d] & TX \times U_\infty \times \mathbb{R}^j \arrow[d] \\ M \arrow[r, "g"] & X \times \mathrm{G}\mathcal{W}(d+1,\infty) \end{tikzcd} \tag{3.6}$$

such that the X-coordinate of g is a proper map $M \to X$. The $\mathbb{R}^j$ factor in the top row, with unspecified j, is there for stabilization purposes. The map $\hat{g}$ should be thought of as a *stable* vector bundle map from $TM \times \mathbb{R}$ to $TX \times U_\infty$, covering g, where U_∞ is the tautological vector bundle of fiber dimension $d+1$ on $\mathrm{G}\mathcal{W}(d+1,\infty)$.

Let now $(\pi, \hat{f}) \in h\mathcal{W}(X)$, where $\hat{f}$ is a section of $J^2(T^\pi E) \to E$ with underlying map $f\colon E \to \mathbb{R}$. See Definition 2.13. After a small deformation which does not affect the concordance class of $(\pi, \hat{f})$, we may assume that f is transverse to $0 \in \mathbb{R}$ (not necessarily fiberwise) and get a manifold $M = f^{-1}(0)$ with $\dim(M) = \dim(X) + d$. The restriction of π to M is a proper map $M \to X$, by the definition of $h\mathcal{W}(X)$. The section $\hat{f}$ yields for each $z \in E$ a map

$$\hat{f}(z) \quad = \quad f(z) + \ell_z + q_z\colon \quad (T^\pi E)_z \to \mathbb{R}$$

with the property that the quadratic term q_z is nondegenerate when the linear term ℓ_z is zero. For $z \in M$ the constant $f(z)$ is zero, so the restriction $T^\pi E|M$ is a $(d+1)$-dimensional vector bundle on M with the extra structure considered in Section 3.1. Thus $T^\pi E|M$ is classified by a map from M to the space $\mathrm{G}\mathcal{W}(d+1,\infty)$: there is a bundle diagram

$$\begin{tikzcd} T^\pi E|M \arrow[r] \arrow[d] & U_\infty \arrow[d] \\ M \arrow[r, "\kappa"] & \mathrm{G}\mathcal{W}(d+1,\infty). \end{tikzcd}$$

Let $g\colon M \longrightarrow X \times \mathrm{G}\mathcal{W}(d+1,\infty)$ be the map $z \mapsto (\pi(z), \kappa(z))$. We now have a canonical vector bundle map

$$\hat{g}\colon TM \times \mathbb{R} \;\cong\; TE|M \;\cong\; \pi^*TX|M \oplus T^\pi E|M \;\longrightarrow\; TX \times U_\infty$$

and we get a triple $(M, g, \hat{g})$ which represents an element of $[X, \Omega^\infty \mathbf{hW}]$ in the bordism-theoretic description. It is easily verified that the bordism class of $(M, g, \hat{g})$ depends only on the concordance class of the pair $(\pi, \hat{f})$. Thus we have defined the map τ of (3.5).

THEOREM 3.5. *The natural map $\tau\colon h\mathcal{W}[X] \to [X, \Omega^\infty \mathbf{hW}]$ is a bijection when X is a closed manifold.*

Proof. We define a map σ in the other direction by running the construction τ backwards. We use the bordism group description (3.6) of $[X, \Omega^\infty \mathbf{hW}]$. Let $(M, g, \hat{g})$ be a representative, with $g\colon M \to X \times \mathrm{G}\mathcal{W}(d+1, \infty)$ and

$$\hat{g}\colon TM \times \mathbb{R} \times \mathbb{R}^j \quad \longrightarrow \quad TX \times U_\infty \times \mathbb{R}^j \ .$$

By obstruction theory, see Lemma 3.6 below, we can suppose that $j = 0$. We write $E = M \times \mathbb{R}$ and $\pi_E\colon E \to X$ for the composition of the projection $E \to M$ with the first component of g. The map $\hat{g}$, now with $j = 0$, has a first component $TM \times \mathbb{R} \to TX$. We (pre-)compose it with the evident vector bundle map from $TE \cong TM \times T\mathbb{R}$ to $TM \times \mathbb{R}$ which covers the projection from $E \cong M \times \mathbb{R}$ to M. The result is a map of vector bundles

$$\hat{\pi}_E\colon TE \longrightarrow TX,$$

covering π_E and surjective in the fibers. Since E is an open manifold, Phillips' submersion theorem [34], [15], [16] applies to show that $(\pi_E, \hat{\pi}_E)$ is homotopic through fiberwise surjective bundle maps to a pair $(\pi, d\pi)$ where $\pi\colon E \to X$ is a submersion and $d\pi\colon TE \to TX$ is its differential.

This homotopy lifts to a homotopy of vector bundle maps which are isomorphic on the fibers, starting with $\hat{g}\colon TE \to TX \times U_\infty$ and ending with a map from TE to $TX \times U_\infty$ which refines the differential $d\pi\colon TE \to TX$. Its restriction to $T^\pi E \subset TE$ is a vector bundle map $T^\pi E \to U_\infty$, still isomorphic on the fibers, which equips each fiber $(T^\pi E)_z$ of $T^\pi E$ with a Morse type map

$$\ell_z + q_z\colon (T^\pi E)_z \to \mathbb{R}.$$

Let $f\colon E \to \mathbb{R}$ be the projection onto the $\mathbb{R}$ factor, and let

$$\hat{f}(z) = f(z) + \ell_z + q_z \in J^2(T^\pi E).$$

The map f is proper, since X and hence M are compact. Consequently the pair $(\pi, \hat{f})$ represents an element in $h\mathcal{W}[X]$. Its concordance class depends only on the bordism class of $(M, g, \hat{g})$; the verification uses a relative version of Lemma 3.6. This describes a map

$$\sigma\colon [X, \Omega^\infty \mathbf{hW}] \longrightarrow h\mathcal{W}[X].$$

It is obvious from the constructions that $\tau \circ \sigma = \mathrm{id}$. In order to evaluate the composition $\sigma \circ \tau$, it suffices by Lemma 2.20 to evaluate it on an element $(\pi, \hat{f})$ where $f\colon E \to \mathbb{R}$ is regular, so that $E \cong M \times \mathbb{R}$ with $M = f^{-1}(0)$. For $(y, r) \in M \times \mathbb{R}$, the map

$$\hat{f}(y, r)\colon (T^\pi(M \times \mathbb{R}))_{(y,r)} \quad \longrightarrow \quad \mathbb{R}$$

is a second degree polynomial of Morse type. The homotopy

$$\hat{f}_t(y, r) = \hat{f}(y, tr) + (1-t)r\,,$$

suitably reparametrized, shows that $(\pi, \hat{f})$ is concordant to $(\pi, \hat{f}_0)$, which represents the image of $(\pi, \hat{f})$ under $\sigma \circ \tau$. Therefore $\sigma \circ \tau = \mathrm{id}$. $\square$

Lemma 3.6. *Let T and U be k-dimensional vector bundles over a manifold M. Let* $\mathrm{iso}(T,U) \to M$ *be the fiber bundle on M whose fiber at $x \in M$ is the space of linear isomorphisms from T_x to U_x. The stabilization map* $\mathrm{iso}(T,U) \to \mathrm{iso}(T \times \mathbb{R}, U \times \mathbb{R})$ *induces a map of section spaces which is* $(k - \dim(M) - 1)$*-connected.*

Proof. We use the following general principle. Suppose that $Y \to M$ and $Y' \to M$ are fibrations and that $f\colon Y \to Y'$ is a map over M. Suppose that for each $x \in M$, the restriction $Y_x \to Y'_x$ of f to the fibers over x is c-connected. Then the induced map of section spaces, $\Gamma(Y) \to \Gamma(Y')$, is $(c-m)$-connected where $m = \dim(M)$.

The proof of this proceeds as follows: Fix $s \in \Gamma(Y')$. The homotopy fiber of $\Gamma(Y) \to \Gamma(Y')$ over s is easily identified with the section space $\Gamma(Y'')$ of another fibration $Y'' \to M$, defined by

$$Y''_x = \mathrm{hofiber}_{s(x)}\,(Y_x \to Y'_x)\,.$$

By assumption each Y''_x is $(c-1)$-connected. Hence by obstruction theory or a simple induction over skeletons, $\Gamma(Y'')$ is $(c-1-m)$-connected. Since this holds for arbitrary s, all homotopy fibers of $\Gamma(Y) \to \Gamma(Y')$ are $(c-1-m)$-connected. Consequently $\Gamma(Y) \to \Gamma(Y')$ is $(c-m)$-connected.

Now for the application: The inclusion $\mathrm{GL}(k) \to \mathrm{GL}(k+1)$ is $(k-1)$-connected. Hence the stabilization map $\mathrm{iso}(T,U) \to \mathrm{iso}(T \times \mathbb{R}, U \times \mathbb{R})$ is $(k-1)$-connected on the fibers, and induces a $((k-1)-m)$-connected map of section spaces. □

The following is an ingredient in a fiberwise version of the Pontryagin-Thom construction which we will need in a moment.

Definition 3.7. Let $p\colon Y \to X$ be a smooth submersion, $C \subset Y$ a smooth submanifold, and suppose that $p|C$ is a submersion. A *vertical* tubular neighborhood for C in Y consists of a smooth vector bundle $q\colon N \to C$ with zero section s, and an open embedding $e\colon N \to Y$ such that $es = \text{inclusion}\colon C \to Y$ and $pe = pq\colon N \to X$.

Now we give a detailed description of a map $|h\mathcal{W}| \to \Omega^\infty \mathbf{hW}$ which induces (3.5). It relies entirely on the Pontryagin-Thom collapse construction.

We begin by describing a variant $h\mathcal{W}^{(r)}$ of $h\mathcal{W}$, depending on an integer $r > 0$. Fix X in $\mathscr{X}$. An element of $h\mathcal{W}^{(r)}(X)$ is a quadruple $(\pi, \hat{f}, w, N)$ where $\pi\colon E \to X$ and $\hat{f}$ are as in Definition 2.13. The remaining data are a smooth embedding

$$w\colon E \;\longrightarrow\; X \times \mathbb{R} \times \mathbb{R}^{d+r}$$

which covers $(\pi, f)\colon E \to X \times \mathbb{R}$, and a *vertical* tubular neighborhood N for the submanifold $w(E)$ of $X \times \mathbb{R} \times \mathbb{R}^{d+r}$, so that the projection $N \to w(E)$ is a map over $X \times \mathbb{R}$. The forgetful map taking an element $(\pi, \hat{f}, w, N)$ to $(\pi, \hat{f})$ is a map of sheaves

$$h\mathcal{W}^{(r)} \longrightarrow h\mathcal{W}$$

on $\mathscr{X}$. This is highly connected if r is large, by Whitney's embedding theorem and the tubular neighborhood theorem, so that the resulting map from $\operatorname{colim}_r h\mathcal{W}^{(r)}$ to $h\mathcal{W}$ is a weak equivalence of sheaves. (The sequential direct limit is formed by sheafifying the "naive" direct limit, which is a presheaf on $\mathscr{X}$. It is easy to verify that passage to representing spaces commutes with sequential direct limits up to homotopy equivalence.)

Let $\mathcal{Z}^{(r)}$ be the sheaf taking an X in $\mathscr{X}$ to the set of maps

$$X \times \mathbb{R} \longrightarrow \Omega^{d+r}\mathrm{Th}\,(U_r^{\perp}).$$

Then the representing space of $\mathcal{Z}^{(r)}$ approximates $\Omega^{\infty}\mathbf{hW}$, in the sense that $\operatorname{colim}_r |\mathcal{Z}^{(r)}| \simeq \Omega^{\infty}\mathbf{hW}$. The Pontryagin-Thom collapse construction gives us a map of sheaves

$$\tau^{(r)}\colon h\mathcal{W}^{(r)} \longrightarrow \mathcal{Z}^{(r)}. \tag{3.7}$$

In detail: let $(\pi, \hat{f}, w, N)$ be an element of $h\mathcal{W}^{(r)}(X)$, where $\hat{f}$ is a section of $J^2(T^{\pi}E) \to E$; see Lemma 3.4. The differential dw determines, for each $z \in E$, a triple $(V_z, \ell_z, q_z) \in \mathrm{G}\mathcal{W}(d+1, r)$. Here V_z is $dw((T^{\pi}E)_z)$, viewed as a subspace of the vertical tangent space at $w(z)$ of the projection

$$X \times \mathbb{R} \times \mathbb{R}^{d+r} \quad \longrightarrow \quad X\,,$$

which we in turn may identify with $\mathbb{R}^{d+1+r}$, and $\ell_z + q_z$ is the nonconstant part of $\hat{f}(z)$. In particular $z \mapsto (V_z, \ell_z, q_z)$ defines a map $\kappa\colon E \to \mathrm{G}\mathcal{W}(d+1, r)$. This extends canonically to a pointed map

$$\mathrm{Th}\,(N) \longrightarrow \mathrm{Th}\,(U_r^{\perp})$$

because N is identified with $\kappa^* U_r^{\perp}$. But $\mathrm{Th}\,(N)$ is a quotient of $X \times \mathbb{R} \times S^{d+r}$ where we regard S^{d+r} as the one-point compactification of $\mathbb{R}^{d+r}$. Thus we have constructed a map

$$X \times \mathbb{R} \times S^{d+r} \longrightarrow \mathrm{Th}\,(U_r^{\perp})$$

or equivalently, a map $X \times \mathbb{R} \longrightarrow \Omega^{d+r}\mathrm{Th}\,(U_r^{\perp})$. Viewed as an element of $\mathcal{Z}^{(r)}(X)$, that map is the image of $(\pi, \hat{f}, w, N)$ under $\tau^{(r)}$ in (3.7). Taking colimits over r, we therefore have a diagram

$$|h\mathcal{W}| \xleftarrow[j]{\simeq} \operatorname{colim}_r |h\mathcal{W}^{(r)}| \longrightarrow \operatorname{colim}_r |\mathcal{Z}^{(r)}| \xrightarrow{\simeq} \Omega^{\infty}\mathbf{hW}$$

which amounts to a map $\tau\colon |h\mathcal{W}| \to \Omega^{\infty}\mathbf{hW}$. (A homotopy inverse i for the map labelled j is unique up to "contractible choice" provided it is chosen together with a homotopy $ji \simeq \mathrm{id}$.)

THEOREM 3.8. *The map $\tau\colon |h\mathcal{W}| \to \Omega^\infty \mathbf{hW}$ is a homotopy equivalence.*

Proof. This follows from Theorem 3.5 and a theorem of J. H. C. Whitehead which tells us that it suffices to check that τ induces isomorphisms on all homotopy groups. The only problem is that Theorem 3.5 is a statement about free (as opposed to based) homotopy classes. However, τ turns out to be a map between spaces with monoid structure (up to homotopy), and in this situation one easily passes between based and unbased homotopy classes. Here are some details. The monoid structure on $|h\mathcal{W}|$ is induced by a monoid structure on $\mathcal{W}$ itself given by "disjoint union":

$$\mathcal{W}(X) \times \mathcal{W}(X) \xrightarrow{\mu} \mathcal{W}(X)\,;\, ((\pi,\hat{f}),(\psi,\hat{g})) \mapsto (\pi \sqcup \psi, \hat{f} \sqcup \hat{g})$$

where the source of $\pi \sqcup \psi$ is the disjoint union of the sources of π and ψ. (See the remark just below.)

To make the monoid structure explicit in the case of the target, we introduce $\mathbf{hW} \vee \mathbf{hW}$ and the corresponding infinite loop space

$$\Omega^\infty(\mathbf{hW} \vee \mathbf{hW}) = \operatorname{colim}_n \Omega^{d+n}\big(\mathrm{Th}\,(U_n^\perp) \vee \mathrm{Th}\,(U_n^\perp)\big).$$

The two maps from $\mathbf{hW} \vee \mathbf{hW}$ to $\mathbf{hW}$ which collapse one of the two wedge summands lead to a weak equivalence $\Omega^\infty(\mathbf{hW}\vee\mathbf{hW}) \simeq \Omega^\infty(\mathbf{hW}) \times \Omega^\infty(\mathbf{hW})$ and the fold map $\mathbf{hW} \vee \mathbf{hW} \to \mathbf{hW}$ induces an addition map

$$\Omega^\infty(\mathbf{hW}) \times \Omega^\infty(\mathbf{hW}) \simeq \Omega^\infty(\mathbf{hW} \vee \mathbf{hW}) \quad \longrightarrow \quad \Omega^\infty(\mathbf{hW}).$$

It is clear that τ can be upgraded to respect the additions. Now Theorem 3.5 with $X = \star$ implies that τ induces a bijection

$$\pi_0|h\mathcal{W}| \longrightarrow \pi_0(\Omega^\infty \mathbf{hW})$$

and consequently that $\pi_0|h\mathcal{W}|$ is a group, since $\pi_0(\Omega^\infty\mathbf{hW})$ is. Next, we use Theorem 3.5 with $X = S^n$. The monoid structures imply the isomorphisms

$$\begin{array}{rcl} \pi_n|h\mathcal{W}| & \cong & [S^n, |h\mathcal{W}|\,]/[\star, |h\mathcal{W}|\,]\,, \\ \pi_n(\Omega^\infty\mathbf{hW}) & \cong & [S^n, \Omega^\infty\mathbf{hW}]/[\star, \Omega^\infty\mathbf{hW}] \end{array}$$

for arbitrary choices of base points. Thus the map τ induces an isomorphism of homotopy groups, and Whitehead's theorem implies that it is a homotopy equivalence, since the spaces in question are CW-spaces. $\square$

Remark 3.9. To avoid set-theoretical problems related to disjoint unions, one should regard μ in the above proof as a map from a certain subsheaf $\mathcal{W}\bar{\times}\mathcal{W}$ of $\mathcal{W} \times \mathcal{W}$ to $\mathcal{W}$. An element $((\pi,\hat{f}),(\psi,\hat{g}))$ of $(\mathcal{W}\times\mathcal{W})(X)$ belongs to $(\mathcal{W}\bar{\times}\mathcal{W})(X)$ if the sources $E(\pi)$ and $E(\psi)$ of π and ψ, respectively, are disjoint. Let μ take $((\pi,\hat{f}),(\psi,\hat{g}))$ to $(\pi\cup\psi, \hat{f}\cup\hat{g})$ with

$$\pi \cup \psi\colon E(\pi) \cup E(\psi) \longrightarrow X\,.$$

Note that the inclusion $\mathcal{W}\bar{\times}\mathcal{W} \longrightarrow \mathcal{W} \times \mathcal{W}$ is a weak equivalence.

The arguments above work in a completely similar fashion to identify $|h\mathcal{V}|$. In fact the map τ in Theorem 3.8 restricts to a map from $|h\mathcal{V}|$ to $\Omega^\infty \mathbf{hV}$ and the analogue of Theorem 3.5 holds. Keeping the letter τ for this restriction, we therefore have

THEOREM 3.10. *The map* $\tau\colon |h\mathcal{V}| \to \Omega^\infty \mathbf{hV}$ *is a homotopy equivalence.*

3.3. *The space* $|h\mathcal{W}_{\mathrm{loc}}|$. We start with a description of $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$ as a bordism group. This is very similar to the description of $[X, \Omega^\infty \mathbf{hW}]$ used in the construction of the map (3.5).

LEMMA 3.11. *For* X *in* $\mathscr{X}$, *the group* $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$ *can be identified with the group of bordism classes of triples* $(M, g, \hat{g})$ *consisting of a smooth* M *without boundary*, $\dim(M) = \dim(X) + d$, *and a vector bundle pullback square*

$$\begin{tikzcd} TM \times \mathbb{R} \times \mathbb{R}^j \arrow[r, "\hat{g}"] \arrow[d] & TX \times U_\infty \times \mathbb{R}^j \arrow[d] \\ M \arrow[r, "g"] & X \times \mathrm{G}\mathcal{W}(d+1, \infty) \end{tikzcd}$$

with $j \gg 0$, *such that the map* $g^{-1}(X \times \Sigma(d+1, \infty)) \to X$ *induced by* g *is proper.*

Proof. The standard bordism group description of the homotopy set $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$ has representatives which are vector bundle pullback squares

$$\begin{tikzcd} TY \times \mathbb{R} \times \mathbb{R}^k \arrow[r, "\hat{g}_Y"] \arrow[d] & TX \times \Sigma(d+1, \infty) \times \mathbb{R}^k \arrow[d] \\ Y \arrow[r, "g_Y"] & X \times \Sigma(d+1, \infty) \end{tikzcd} \tag{3.8}$$

for some $k \gg 0$, where the map $Y \to X$ determined by g_Y is proper, $\partial Y = \emptyset$ and $\dim(Y) = \dim(X) - 1$. See Lemma 3.1. We produce reciprocal maps relating this bordism group to the one in Lemma 3.11.

We first identify $U_\infty | \Sigma(d+1, \infty)$ with its dual using the canonical quadratic form q, and then with the normal bundle N of $\Sigma(d+1, \infty)$ in $\mathrm{G}\mathcal{W}(d+1, \infty)$. Let $(M, g, \hat{g})$ be a triple as above, Lemma 3.11. We may assume that g is transverse to $X \times \Sigma(d+1, \infty)$. Then $Y = g^{-1}(X \times \Sigma(d+1, \infty))$ is a smooth submanifold of M, of codimension $d+1$, with normal bundle N_Y. Restriction of g and $\hat{g}$ yields a vector bundle pullback square

$$\begin{tikzcd} (TY \oplus N_Y) \times \mathbb{R} \times \mathbb{R}^j \arrow[r] \arrow[d] & TX \times N \times \mathbb{R}^j \arrow[d] \\ Y \arrow[r] & X \times \Sigma(d+1, \infty)\,. \end{tikzcd}$$

But since N_Y is also identified with the pullback of N, this amounts to a vector bundle pullback square as in (3.8).

Conversely, given data Y, g_Y and $\hat{g}_Y$ as in (3.8), let M be the (total space of the) pullback of N to Y. There is a canonical map $M \to N \subset \mathrm{G}\mathcal{W}(d{+}1,\infty)$, and another from M to X, hence a map $g\colon M \to X \times \mathrm{G}\mathcal{W}(d{+}1,\infty)$. Moreover $\hat{g}_Y$ determines the $\hat{g}$ in a triple $(M, g, \hat{g})$ as above. It is easy to verify that the two maps of bordism groups so constructed are well defined and that they are reciprocal isomorphisms. $\square$

We now turn to the construction of a localized version of (3.5), namely, a natural map

$$\tau_{\mathrm{loc}}\colon h\mathcal{W}_{\mathrm{loc}}[X] \longrightarrow [X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]. \tag{3.9}$$

Let $(\pi, \hat{f}) \in h\mathcal{W}_{\mathrm{loc}}(X)$, where $\pi\colon E \to X$ is a submersion with $(d+1)$-dimensional fibers and $\hat{f}$ is a section of $J^2(T^\pi E) \to E$ with underlying map $f\colon E \to \mathbb{R}$. See Definitions 2.14 and 3.4. We may assume that f is transverse to 0 and get a manifold $M = f^{-1}(0)$. Proceeding exactly as in the construction of the map (3.5), we can promote this to a triple $(M, g, \hat{g})$ where $(g, \hat{g})$ is a vector bundle pullback square

$$\begin{tikzcd} TM \times \mathbb{R} \times \mathbb{R}^j \arrow[r, "\hat{g}"] \arrow[d] & TX \times U_\infty \times \mathbb{R}^j \arrow[d] \\ M \arrow[r, "g"] & X \times \mathrm{G}\mathcal{W}(d+1,\infty)\,. \end{tikzcd}$$

This time, however, we cannot expect that the X-component of g, which is $\pi|M$, is proper. But its restriction to

$$g^{-1}(X \times \Sigma(d+1,\infty)) = \Sigma(\pi, \hat{f}) \cap M$$

is proper, thanks to condition (ia) in Definition 2.14. Therefore $(M, g, \hat{g})$ represents an element in $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$. This is the image of $(\pi, \hat{f})$ under τ_{loc}.

Theorem 3.12. *For compact X in $\mathscr{X}$, the map τ_{loc} from $h\mathcal{W}_{\mathrm{loc}}[X]$ to $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$ is a bijection.*

Proof. There is a map σ_{loc} in the other direction. The construction of σ_{loc} is analogous to that of σ in the proof of Theorem 3.5. It is clear that $\tau_{\mathrm{loc}} \circ \sigma_{\mathrm{loc}}$ is the identity. The verification of $\sigma_{\mathrm{loc}} \circ \tau_{\mathrm{loc}} = \mathrm{id}$ uses Lemma 3.13 below. $\square$

Lemma 3.13. *Let $(\pi, \hat{f}) \in h\mathcal{W}_{\mathrm{loc}}(X)$, with $\pi\colon E \to X$. Let U be an open neighborhood of $\Sigma(\pi, \hat{f})$ in E. Then $(\pi|U, \hat{f}|U) \in h\mathcal{W}_{\mathrm{loc}}(X)$ is concordant to $(\pi, \hat{f})$.*

Proof. The concordance is an element $(\pi^\sharp, \hat{f}^\sharp)$ in $h\mathcal{W}_{\mathrm{loc}}(X \times \mathbb{R})$. Let $E^\sharp \subset E \times \mathbb{R}$ be the union of $E\times\,]-\infty, 1/2[$ and $U \times \mathbb{R}$. Let $\pi^\sharp(z,t) = (\pi(z), t)$ and $\hat{f}^\sharp(z,t) = (\hat{f}(z), t)$ for $(z,t) \in E^\sharp$. Some renaming of the elements of $E^\sharp$ is required to ensure that $\pi^\sharp$ be graphic. □

Next we give a short description of a map $|h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$ which induces (3.9). This is analogous to the construction of the map named τ in Theorem 3.8.

Fix an integer $r > 0$ and X in $\mathscr{X}$. To the data $(\pi, \hat{f})$ in Definition 2.14, with $\pi\colon E \to X$ and $f\colon E \to \mathbb{R}$, we add the following: a smooth embedding

$$w\colon E \quad\longrightarrow\quad X \times \mathbb{R} \times \mathbb{R}^{d+r}$$

which covers $(\pi, f)\colon E \to X \times \mathbb{R}$, a vertical tubular neighborhood N for the submanifold $w(E)$ of $X \times \mathbb{R} \times \mathbb{R}^{d+r}$, and a smooth function $\psi\colon E \to [0,1]$ such that $\psi(z) = 1$ for all $z \in \Sigma(\pi, \hat{f})$. We require that the restriction of $(\pi, f)\colon E \to X \times \mathbb{R}$ to the support of ψ be a proper map.

Making X into a variable now, we can interpret the forgetful map taking $(\pi, \hat{f}, w, N, \psi)$ to $(\pi, \hat{f})$ as a map of sheaves

$$h\mathcal{W}^{(r)}_{\mathrm{loc}} \longrightarrow h\mathcal{W}_{\mathrm{loc}}$$

on $\mathscr{X}$. This map is highly connected if r is large. Let $\mathcal{Z}^{(r)}_{\mathrm{loc}}$ be the sheaf taking an X in $\mathscr{X}$ to the set of maps

$$X \times \mathbb{R} \longrightarrow \Omega^{d+r}\mathrm{cone}\Big(\mathrm{Th}\,(U_r^\perp | \mathrm{G}\mathcal{V}(d+1,r)) \hookrightarrow \mathrm{Th}\,(U_r^\perp)\Big)\,.$$

Here the cone is a reduced mapping cone, regarded as a quotient of a subspace of

$$\mathrm{Th}\,(U_r^\perp) \times [0,1]\,,$$

with $\mathrm{Th}\,(U_r^\perp) \times \{1\}$ corresponding to the base of the cone. The Pontryagin-Thom collapse construction gives us a map of sheaves

$$\tau^{(r)}_{\mathrm{loc}}\colon h\mathcal{W}^{(r)}_{\mathrm{loc}} \longrightarrow \mathcal{Z}^{(r)}_{\mathrm{loc}}. \tag{3.10}$$

In detail: let $(\pi, \hat{f}, w, N, \psi)$ be an element of $h\mathcal{W}^{(r)}_{\mathrm{loc}}(X)$. We assume that $\hat{f}$ is a section of $J^2(T^\pi E) \to E$; see 3.4. The differential dw determines, for each $z \in E$, a triple $(V_z, \ell_z, q_z) \in \mathrm{G}\mathcal{W}(d+1, r)$, as in the proof of Theorem (3.8). This gives us a map

$$\kappa\colon E \to \mathrm{G}\mathcal{W}(d+1,r) \times [0,1]\,,$$

with first coordinate determined by dw and second coordinate equal to ψ. The map κ fits into a vector bundle pullback square

$$\begin{tikzcd} N \arrow[r, "\hat{\kappa}"] \arrow[d] & U_r^\perp \times [0,1] \arrow[d] \\ E \arrow[r, "\kappa"] & \mathrm{G}\mathcal{W}(d+1,r) \times [0,1]. \end{tikzcd}$$

Now we obtain a map from $X \times \mathbb{R} \times S^{d+r}$ to the mapping cone

$$\mathrm{cone}\Big(\mathrm{Th}\,(U_r^{\perp}|\mathrm{G}\mathcal{V}(d+1,r)) \hookrightarrow \mathrm{Th}\,(U_r^{\perp})\Big),$$

viewed as a subquotient of $\mathrm{Th}\,(U_r^{\perp}) \times [0,1]$, by $z \mapsto \hat{\kappa}(z)$ for $z \in N$ and $z \mapsto \star$ for $z \notin N$. It can also be written in the form

$$X \times \mathbb{R} \longrightarrow \Omega^{d+r}\mathrm{cone}\big(\mathrm{Th}\,(U_r^{\perp}|\mathrm{G}\mathcal{V}(d+1,r)) \hookrightarrow \mathrm{Th}\,(U_r^{\perp})\big)$$

so that it is an element of $\mathcal{Z}^{(r)}_{\mathrm{loc}}(X)$. This defines the map $\tau^{(r)}_{\mathrm{loc}}$. Taking colimits over r, we therefore have a diagram

$$|h\mathcal{W}_{\mathrm{loc}}| \xleftarrow{\ \simeq\ } \mathrm{colim}_r\,|h\mathcal{W}^{(r)}_{\mathrm{loc}}| \longrightarrow \mathrm{colim}_r\,|\mathcal{Z}^{(r)}_{\mathrm{loc}}| \xrightarrow{\ \simeq\ } \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$$

which amounts to a map $\tau_{\mathrm{loc}}\colon |h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$. The following is a straightforward consequence of Theorem 3.12 (cf. the proof of Theorem 3.8):

THEOREM 3.14. *The map $\tau_{\mathrm{loc}}\colon |h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$ is a homotopy equivalence.* □

The combination of Theorems 3.14, 3.8, 3.10 and Proposition 3.3 amounts to a proof of Theorem 1.4 from the introduction.

Remark 3.15. We are left with the task of saying exactly how the lower row of diagram (1.9) should be regarded as a homotopy fiber sequence. Define a sheaf $h\mathcal{V}_{\mathrm{loc}}$ on $\mathscr{X}$ by copying Definition 2.12, the definition of $h\mathcal{V}$, but leaving out condition (i). Then $|h\mathcal{V}_{\mathrm{loc}}|$ is contractible by an application of Proposition 2.17. Any choice of nullhomotopy for the inclusion $|h\mathcal{V}| \to |h\mathcal{V}_{\mathrm{loc}}|$ determines a nullhomotopy for $|h\mathcal{V}| \to |h\mathcal{W}_{\mathrm{loc}}|$, since $|h\mathcal{V}_{\mathrm{loc}}| \subset| h\mathcal{W}_{\mathrm{loc}}|$. A nullhomotopy for $|h\mathcal{V}| \to |h\mathcal{W}_{\mathrm{loc}}|$ constructed like that is understood in Theorem 1.4.

3.4. *The space $|\mathcal{W}_{\mathrm{loc}}|$.* The goal is to prove Theorem 1.2, i.e., to show that the inclusion of $\mathcal{W}_{\mathrm{loc}}$ in $h\mathcal{W}_{\mathrm{loc}}$ is a weak equivalence. We begin with the observation that the analogue of Lemma 3.13 holds for $\mathcal{W}_{\mathrm{loc}}$:

LEMMA 3.16. *Let $(\pi, f) \in \mathcal{W}_{\mathrm{loc}}(X)$, with $\pi\colon E \to X$. Let U be an open neighborhood of $\Sigma(\pi, f)$ in E. Then $(\pi|U, f|U) \in \mathcal{W}_{\mathrm{loc}}(X)$ is concordant to (π, f).* □

COROLLARY 3.17. *For X in $\mathscr{X}$, there are natural bijections between the set $\mathcal{W}_{\mathrm{loc}}[X]$ and either of the two sets below*:

(i) *The set of bordism classes of triples (Σ, p, g), where Σ is a smooth manifold without boundary, $p\colon \Sigma \to X \times \mathbb{R}$ is a proper smooth map whose*

X-coordinate $\Sigma \to X$ is an étale map (= local diffeomorphism), and g is a map from Σ to $\Sigma(d+1,\infty)$;

(ii) *The set of bordism classes of triples (Σ_0, v, c) where Σ_0 is a smooth manifold without boundary, $v\colon \Sigma_0 \to X$ is a proper smooth codimension 1 immersion with oriented normal bundle and c is a map from Σ_0 to $\Sigma(d+1,\infty)$.*

The bordism relation in both cases involves certain maps to $X \times [0,1]$: étale maps in the case of (i), codimension one immersions in the case of (ii).

Proof. An element (π, f) of $\mathcal{W}_{\mathrm{loc}}(X)$ determines by Lemma 2.8 a triple (Σ, p, g) as in (i), where Σ is $\Sigma(\pi, f)$ and $p(z) = (\pi(z), f(z))$ for $z \in \Sigma \subset E$. The map g classifies the vector bundle $T^\pi E|\Sigma$ together with the nondegenerate quadratic form determined by (one-half) the fiberwise Hessian of f. Conversely, given a triple (Σ, p, g) we can make an element (π, f) in $\mathcal{W}_{\mathrm{loc}}(X)$. Namely, let $E \to \Sigma$ be the $(d+1)$-dimensional vector bundle classified by g, with the canonical quadratic form $q\colon E \to \mathbb{R}$. Let $(\pi, f)\colon E \to X \times \mathbb{R}$ agree with $q + \bar{p}$, where $\bar{p}$ denotes the composition of the vector bundle projection $E \to \Sigma$ with $p\colon \Sigma \to X \times \mathbb{R}$. The resulting maps from $\mathcal{W}_{\mathrm{loc}}[X]$ to the bordism set in (i), and from the bordism set in (i) to $\mathcal{W}_{\mathrm{loc}}[X]$, are inverses of one another: One of the compositions is obviously an identity, the other is an identity by Lemma 3.16.

Next we relate the bordism set in (i) to that in (ii). A triple (Σ, p, g) as in (i) gives rise to a triple (Σ_0, v, c) as in (ii) provided p is transverse to $X \times 0$. In that case we set $\Sigma_0 = p^{-1}(X \times 0)$ and define v and c as the restrictions of p and g, respectively. Conversely, a triple (Σ_0, v, c) as in (ii) does of course determine a triple (Σ, p, g) as in (i) with $\Sigma = \Sigma_0 \times \mathbb{R}$. The resulting maps from the bordism set in (i) to that in (ii), and vice versa, are inverses of one another: One of the compositions is obviously an identity, the other is an identity by a shrinking lemma analogous to (but easier than) Lemma 2.20. ☐

It is well-known that the bordism set (ii) in Corollary 3.17 is in natural bijection with

$$[X, \Omega^\infty S^{1+\infty}(\Sigma(d+1,\infty)_+)] \quad \cong \quad [X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}].$$

Indeed, Pontryagin-Thom theory allows us to represent elements of the homotopy set $[X, \Omega^\infty S^{1+\infty}(\Sigma(d+1,\infty)_+)]$ by quadruples $(\Sigma_0, v, \hat{v}, c)$ where Σ_0 is smooth without boundary, $\dim(\Sigma_0) = \dim(X) - 1$, the maps v and $\hat{v}$ constitute a vector bundle pullback square

$$\begin{tikzcd}
T\Sigma_0 \times \mathbb{R} \times \mathbb{R}^j \arrow[r, "\hat{v}"] \arrow[d] & TX \times \mathbb{R}^j \arrow[d] \\
\Sigma_0 \arrow[r, "v"] & X
\end{tikzcd}$$

(for some $j \gg 0$) with proper v, and c is any map from Σ_0 to $\Sigma(d+1,\infty)$. By Lemma 3.6 we can take $j = 0$ and by immersion theory [42], [18], [16] we can assume $\hat{v} = dv$, that is, v is an immersion and $\hat{v}$ is its (total) differential.

Consequently $\mathcal{W}_{\mathrm{loc}}[X]$ is in natural bijection with $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$. It is easy to verify that this natural bijection is induced by the composition

$$|\mathcal{W}_{\mathrm{loc}}| \hookrightarrow |h\mathcal{W}_{\mathrm{loc}}| \xrightarrow{\tau_{\mathrm{loc}}} \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$$

where τ_{loc} is the map of (3.10), (3.9) and Theorem 3.14. We conclude that the composition is a homotopy equivalence (cf. the proof of Theorem 3.8). Since τ_{loc} itself is a homotopy equivalence, it follows that the inclusion $|\mathcal{W}_{\mathrm{loc}}| \hookrightarrow |h\mathcal{W}_{\mathrm{loc}}|$ is a homotopy equivalence. This is Theorem 1.3 from the introduction.

4. Application of Vassiliev's h-principle

This section contains the proof of Theorem 1.2. It is based upon a special case of Vassiliev's *first main theorem*, [45, ch.III] and [46].

Let $\mathfrak{A} \subset J^2(\mathbb{R}^r, \mathbb{R})$ denote the space of 2-jets represented by $f\colon (\mathbb{R}^r, z) \to \mathbb{R}$ with $f(z) = 0$, $df(z) = 0$ and $\det(d^2 f(z)) = 0$, where $d^2 f(z)$ denotes the Hessian. This set has codimension $r+2$ and is invariant under diffeomorphisms $\mathbb{R}^r \to \mathbb{R}^r$.

Let N^r be a smooth compact manifold with boundary and let $\psi\colon N \to \mathbb{R}$ be a fixed smooth function with $j^2\psi(z) \notin \mathfrak{A}$ for z in a neighborhood of the boundary. (Use local coordinates near z. The condition means that near ∂N, all singularities of ψ with value 0 are of Morse type, i.e., nondegenerate.) Define spaces

$$\begin{aligned} \Phi(N, \mathfrak{A}, \psi) &= \{ f \in C^\infty(N, \mathbb{R}) \mid f = \psi \text{ near } \partial N,\ j^2 f(z) \notin \mathfrak{A} \text{ for } z \in N \}, \\ h\Phi(N, \mathfrak{A}, \psi) &= \{ \hat{f} \in \Gamma J^2(N, \mathbb{R}) \mid \hat{f} = j^2\psi \text{ near } \partial N,\ \hat{f}(z) \notin \mathfrak{A} \text{ for } z \in N \}, \end{aligned}$$

where $\Gamma J^2(N, \mathbb{R})$ denotes the space of smooth sections of $J^2(N, \mathbb{R}) \to N$. Both are equipped with the standard C^∞ topology. The special case of Vassiliev's theorem that we need is the statement that the map

$$j^2\colon \Phi(N, \mathfrak{A}, \psi) \longrightarrow h\Phi(N, \mathfrak{A}, \psi) \tag{4.1}$$

induces an isomorphism in cohomology with arbitrary untwisted coefficients. (Equivalently by the universal coefficient theorem, it induces an isomorphism in integral homology.)

We briefly indicate how (4.1) relates to the jet prolongation map from $|\mathcal{W}|$ to $|h\mathcal{W}|$ or equivalently (by Lemma 2.23) to the map $|\mathcal{W}^0| \to | h\mathcal{W}^0|$. Let (N, ψ) be as above with $\dim(N) = d + 1$. We assume in addition that $\psi(N) \subset A$ and $\psi(\partial N) \subset \partial A$, where $A \subset \mathbb{R}$ is a compact interval with $0 \in \mathrm{int}(A)$, and that ψ is nonsingular near ∂A. For X in $\mathscr{X}$, let $\mathcal{W}^0_\psi(X) \subset \mathcal{W}^0(X)$ consist of the pairs (π, f) as in 2.21, with $\pi\colon E \to X$, such that E contains an embedded copy of

$N \times X$, the map f agrees with $(z,x) \mapsto \psi(z)$ on a neighborhood of $\partial N \times X$ in $N \times X$, and $f^{-1}(0) \subset N \times X$. Restricting f to $N \times X$ defines a map from $\mathcal{W}^0_\psi(X)$ to the set of smooth maps $X \to \varPhi(N,\psi,\mathfrak{A})$. Making X into a variable, we have a map of sheaves which easily leads to a weak homotopy equivalence

$$|\mathcal{W}^0_\psi| \simeq \varPhi(N,\psi,\mathfrak{A}).$$

Analogous definitions, with $\mathcal{W}^0$ replaced by $h\mathcal{W}^0$ and ψ by its jet prolongation $j^2_\pi\psi$, lead to a weak homotopy equivalence

$$|h\mathcal{W}^0_\psi| \simeq h\varPhi(N,\psi,\mathfrak{A}).$$

Arranging these two homotopy equivalences in a commutative square, we deduce from (4.1) that the jet prolongation map $|\mathcal{W}^0_\psi| \to | h\mathcal{W}^0_\psi|$ is a homology equivalence.

Given an element $(\pi,f) \in \mathcal{W}^0(X)$ with $\pi\colon E \to X$, it is of course not always possible to find a pair (N,ψ) and an embedding $N \times X \to E$ over X with the good properties above. However, the problem can always be solved locally. Namely, each $x \in X$ has an open neighborhood U in X such that $\pi^{-1}(U)$ admits such an embedding, $N \times U \to \pi^{-1}(U)$, for suitable (N,ψ). This fact, its analogue for the sheaf $h\mathcal{W}^0$ and a general gluing technique, developed in Section 4.1 below, allow us then to conclude that $|\mathcal{W}^0| \to | h\mathcal{W}^0|$ induces an isomorphism in homology.

4.1. *Sheaves with category structure.* Our goal here is to develop an abstract gluing principle, summarized in Proposition 4.6 and relying on Definition 4.1. It is a translation into the language of sheaves of something which homotopy theorists are very familar: the homotopy invariance property of homotopy colimits. See Section B.2 for background and motivation. Since it is relatively easy to reduce the homotopy colimit concept to the classifying space construction for categories, our translation effort begins with a discussion of sheaves taking values in the category of small categories, and a "classifying sheaf" construction for such sheaves.

Let $\mathcal{F}\colon \mathscr{X} \to \mathscr{C}at$ be a sheaf with values in small categories. Taking nerves defines a sheaf with values in the category of simplicial sets,

$$N_\bullet\mathcal{F}\colon \mathscr{X} \to \mathscr{S}ets_\bullet$$

with $N_0\mathcal{F} = \mathrm{ob}(\mathcal{F})$ the sheaf of objects and $N_1\mathcal{F} = \mathrm{mor}(\mathcal{F})$ the sheaf of morphisms. We have the associated bisimplicial set $N_\bullet\mathcal{F}(\Delta^\bullet_e)$ and recall [36] that the realization of its diagonal is homeomorphic to either of its double realizations,

$$\begin{aligned} |\,\underline{k} \mapsto N_k\mathcal{F}(\Delta^k_e)\,| \;&\cong\; \big|\,\underline{\ell} \mapsto |\,\underline{k} \mapsto N_k\mathcal{F}(\Delta^\ell_e)\,|\big| \;=\; \big|\,\underline{\ell} \mapsto B(\mathcal{F}(\Delta^\ell_e))\,\big| \\ &\cong\; \big|\,\underline{k} \mapsto |\,\underline{\ell} \mapsto N_k\mathcal{F}(\Delta^\ell_e)\,|\big| \;=\; \big|\,\underline{k} \mapsto |N_k\mathcal{F}|\big|. \end{aligned} \tag{4.2}$$

There is a topological category $|\mathcal{F}|$ with object space $|N_0\mathcal{F}|$ and morphism space $|N_1\mathcal{F}|$. (To be quite precise, $|\mathcal{F}|$ is a category object in the category of compactly generated Hausdorff spaces.) Since $|N_k\mathcal{F}| = N_k|\mathcal{F}|$ by A.3, the last of the five expressions in (4.2) is the classifying space $B|\mathcal{F}|$ of the topological category $|\mathcal{F}|$.

We next give another construction of $B|\mathcal{F}|$ related to Steenrod's coordinate bundles (i.e., bundles viewed as 1-cocycles). We shall consider locally finite open covers $\mathscr{Y} = (Y_j)_{j\in J}$ of spaces X in $\mathscr{X}$, indexed by a *fixed* infinite set J. The local finiteness condition means that each $x \in X$ has a neighborhood U such that $\{j \in J \mid Y_j \cap U \neq \emptyset\}$ is a finite subset of J. We use a fixed indexing set J, independent of X in $\mathscr{X}$, to ensure good gluing properties: suppose that X is the union of two open subsets, $X = X' \cup X''$, with intersection $A = X' \cap X''$, and that $(Y'_j)_{j\in J}$ and $(Y''_j)_{j\in J}$ are open coverings of X' and X'', respectively. The coverings agree on A if $Y'_j \cap A = Y''_j \cap A$ for all $j \in J$. In that case, $(Y'_j \cup Y''_j)_{j\in J}$ is an open covering of X which *induces* the open coverings $(Y'_j)_{j\in J}$ and $(Y''_j)_{j\in J}$ of X' and X'', respectively.

For each finite nonempty subset $S \subset J$ we write

$$Y_S = \bigcap_{j\in S} Y_j\,.$$

Associated to the cover $\mathscr{Y}$ there is a topological category, denoted $X_{\mathscr{Y}}$ in [41, §4], with

$$\mathrm{ob}(X_{\mathscr{Y}}) = \coprod_S Y_S\,, \qquad \mathrm{mor}(X_{\mathscr{Y}}) = \coprod_R \coprod_{S\supset R} Y_S\,,$$

the source map given by the identities $Y_S \to Y_S$ and the target map given by the inclusions $Y_S \to Y_R$ for $S \supset R$. A continuous functor from $X_{\mathscr{Y}}$ to a topological group G, viewed as a topological category with one object, is equivalent to a collection of maps

$$\varphi_{RS}\colon Y_S \longrightarrow G\ ,$$

one for each pair $R \subset S$ of finite subsets of J, subject to certain "cocycle" conditions expressing the fact that the functor preserves compositions. The cocycle conditions are listed in Definition 4.1 below, but in the more general setting where the group of maps from Y_S to G is replaced by the category $\mathcal{F}(Y_S)$.

Definition 4.1. For X in $\mathscr{X}$ an element of $\beta\mathcal{F}(X)$ is a pair $(\mathscr{Y}, \varphi_{\bullet\bullet})$ where $\mathscr{Y}$ is a locally finite open cover of X, indexed by J, and $\varphi_{\bullet\bullet}$ associates to each pair of finite, nonempty subsets $R \subset S$ of J a morphism $\varphi_{RS} \in N_1\mathcal{F}(Y_S)$ subject to the following cocycle conditions:

(i) Every φ_{RR} is an identity morphism;

(ii) For $R \subset S \subset T$, we have $\varphi_{RT} = (\varphi_{RS}|Y_T) \circ \varphi_{ST}$.

Condition (ii) includes the condition that the right-hand composition is *defined*; in particular, taking $S = T$ one finds that the source of φ_{RS} is the object φ_{SS}, and taking $R = S$ one finds that the target of φ_{ST} is $\varphi_{SS}|Y_T$.

The sets $\beta\mathcal{F}(X)$ define a sheaf $\beta\mathcal{F}\colon \mathscr{X} \to \mathscr{S}ets$ and hence a space $|\beta\mathcal{F}|$. The following key theorem is one of our main tools used in the proof of both Theorem 1.2 and Theorem 1.5. Its proof is deferred to Appendix A.

THEOREM 4.2. *The spaces $|\beta\mathcal{F}|$ and $B|\mathcal{F}|$ are homotopy equivalent.*

Consider the example where $\mathcal{F}(X)$ is the set of continuous maps from X to a topological group G, made into a group by pointwise multiplication. An element $(\mathscr{Y}, \varphi_{\bullet\bullet})$ of $\beta\mathcal{F}(X)$ is a collection of gluing data for a principal G-bundle $P \to X$ with chosen trivializations over each Y_R. Namely,

$$P = \coprod_R \{R\} \times Y_R \times G \Big/ \sim$$

where R runs through the finite nonempty subsets of J, and the equivalence relation identifies (R, x, g_1) with (S, x, g_2) if $R \subset S$ and $\varphi_{RS}(x)g_2 = g_1$.

The topological category $|\mathcal{F}|$ is a topological group and comes with a continuous homomorphism $|\mathcal{F}| \to G$ which is clearly a weak homotopy equivalence. So $B|\mathcal{F}| \simeq BG$. Thus Theorem 4.2 reduces to the well-known statement that concordance classes of principal Steenrod G-bundles are classified by BG.

Consider next the case where $\mathcal{F}(X) = \mathrm{map}(X, \mathscr{C})$ for a small topological category $\mathscr{C}$. That is, $\mathrm{ob}(\mathcal{F}(X))$ and $\mathrm{mor}(\mathcal{F}(X))$ are the sets of continuous maps from X to $\mathrm{ob}(\mathscr{C})$ and $\mathrm{mor}(\mathscr{C})$, respectively. Then an element of $\beta(\mathcal{F}(X))$ is a covering $\mathscr{Y}$ of X together with a continuous functor from $X_{\mathscr{Y}}$ to $\mathscr{C}$. If $k \mapsto N_k\mathscr{C}$ is a good simplicial space in the sense of [39], then the canonical map $B|\mathcal{F}| \to B\mathscr{C}$ is a weak equivalence since it is induced by weak equivalences $N_k|\mathcal{F}| \cong |N_k\mathcal{F}| \to N_k\mathscr{C}$. Therefore Theorem 4.2 applied to this situation implies that homotopy classes of maps $X \to B\mathscr{C}$ are in natural bijection with concordance classes of pairs consisting of a covering $\mathscr{Y}$ and a continuous functor from $X_{\mathscr{Y}}$ to $\mathscr{C}$. This statement may have folklore status. It appears explicitly in lectures given by tom Dieck in 1972, but it seems that tom Dieck attributes it to Segal. (We are indebted to R. Vogt who kindly sent us copies of a few pages of lecture notes taken by himself at the time.) Moerdijk has developed this theme much further in [30].

In our applications of Theorem 4.2, the categories $\mathcal{F}(X)$ will typically be partially ordered sets or will have been obtained from a functor

$$\mathcal{F}_\bullet\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \text{ sheaves on } \mathscr{X}\,,$$

where $\mathscr{C}$ is a small category. Given such a functor one can define a category valued sheaf $\mathscr{C}^{\mathrm{op}}\!\int\!\mathcal{F}_\bullet$ on $\mathscr{X}$. Its value on a *connected* manifold X is the category whose objects are pairs (c, ω) with $c \in \mathrm{ob}(\mathscr{C})$, $\omega \in \mathcal{F}_c(X)$ and where a morphism $(b,\tau) \to (c,\omega)$ is a morphism $f\colon b \to c$ in $\mathscr{C}$ with $f^*(\omega) = \tau$. Then

$$|\beta(\mathscr{C}^{\mathrm{op}}\textstyle\int\mathcal{F}_\bullet)| \;\simeq\; B|\mathscr{C}^{\mathrm{op}}\textstyle\int\mathcal{F}_\bullet| \;\simeq\; \operatorname*{hocolim}_{c\in\mathscr{C}} |\mathcal{F}_c|$$

(see §B.2 for details).

Definition 4.3. The sheaf $\beta(\mathscr{C}^{\mathrm{op}}\int\mathcal{F}_\bullet)\colon \mathscr{X} \longrightarrow \mathscr{C}at$ will also be written

$$\operatorname*{hocolim}_{c\in\mathscr{C}} \mathcal{F}_c\,.$$

Spelled out, an element of $(\mathrm{hocolim}_c\,\mathcal{F}_c)(X)$ consists of

(i) a covering $\mathscr{Y}$ of X indexed by J,

(ii) a functor θ from the poset of pairs (S,z), where $S \subset J$ is finite nonempty and $z \in \pi_0(Y_S)$, to $\mathscr{C}$,

(iii) and finally elements $\omega_{S,z} \in \mathcal{F}_{\theta(S,z)}(Y_{S,z})$, where $Y_{S,z}$ denotes the connected component of Y_S corresponding to $z \in \pi_0(Y_S)$. The elements $\omega_{S,z}$ are related to each other via the maps

$$\mathcal{F}_{\theta(T,z)}(Y_{T,z}) \longrightarrow \mathcal{F}_{\theta(S,\bar z)}(Y_{T,z}) \longleftarrow \mathcal{F}_{\theta(S,\bar z)}(Y_{S,\bar z})$$

for each $S \subset T$ and $z \in \pi_0(Y_T)$ with image $\bar z \in \pi_0(Y_S)$.

We close with an application of Theorem 4.2 which will be used below to extend the special case of Vassiliev's theorem mentioned earlier.

Definition 4.4. Let $\mathcal{E},\mathcal{F}\colon \mathscr{X} \to \mathscr{C}at$ be sheaves and $g\colon \mathcal{E} \to \mathcal{F}$ a map between them. We say that g is a *transport projection*, or that it has the unique lifting property for morphisms, if the following square is a pullback square of sheaves on $\mathscr{X}$:

$$\begin{tikzcd} N_1\mathcal{E} \arrow[r, "d_0"] \arrow[d, "g"] & N_0\mathcal{E} \arrow[d, "g"] \\ N_1\mathcal{F} \arrow[r, "d_0"] & N_0\mathcal{F} \end{tikzcd}$$

where d_0 is the source operator.

Definition 4.5. A natural transformation $u\colon \mathcal{F} \to \mathcal{G}$ of sheaves on $\mathscr{X}$ has the *concordance lifting property* if, for X in $\mathscr{X}$ and $s \in \mathcal{F}(X)$, any concordance $h \in \mathcal{G}(X \times \mathbb{R})$ starting at $u(s)$ lifts to a concordance $H \in \mathcal{F}(X \times \mathbb{R})$ starting at s.

Let $g\colon \mathcal{E} \to \mathcal{F}$ be a map of set-valued sheaves on $\mathscr{X}$. An element $a \in \mathcal{F}(\star)$ gives rise to an element again denoted $a \in \mathcal{F}(X)$ for each $X \in \mathscr{X}$. The *fiber* of g over a is the sheaf $\mathcal{E}_a$ defined by

$$\mathcal{E}_a(X) = \{s \in \mathcal{E}(X) \mid g(s) = a\}.$$

Proposition 4.6. *Let $g\colon \mathcal{E} \to \mathcal{F}$ and $g'\colon \mathcal{E}' \to \mathcal{F}$ be transport projections and let $u\colon \mathcal{E} \to \mathcal{E}'$ be a map of sheaves over $\mathcal{F}$ which respects the category structures. Suppose that the maps $N_0\mathcal{E} \to N_0\mathcal{F}$ and $N_0\mathcal{E}' \to N_0\mathcal{F}$ obtained from g and g' have the concordance lifting property and that, for each object a of $\mathcal{F}(\star)$, the restriction $N_0\mathcal{E}_a \to N_0\mathcal{E}'_a$ of u to the fibers over a is a weak equivalence (resp. induces an integral homology equivalence of the representing spaces). Then $\beta u\colon \beta\mathcal{E} \to \beta\mathcal{E}'$ is a weak equivalence (resp. induces an integral homology equivalence of the representing spaces).*

Proof. According to Theorem 4.2 it suffices to prove that u induces a homotopy (homology) equivalence from $B|\mathcal{E}|$ to $B|\mathcal{E}'|$. By (4.2) and Lemma B.1 it is then also enough to show that

$$N_k(u)\colon N_k\mathcal{E} \longrightarrow N_k\mathcal{E}'$$

becomes a homotopy equivalence (homology equivalence) after passage to representing spaces, for each $k \geq 0$. We note that the simplicial spaces obtained from a bisimplicial set by realizing in either direction are good in the sense of [39].

Since g and g' are transport projections, an obvious inductive argument shows that, for each k, the diagrams

$$\begin{tikzcd} N_k\mathcal{E} \arrow[r] \arrow[d, "g"] & N_0\mathcal{E} \arrow[d, "g"] \\ N_k\mathcal{F} \arrow[r] & N_0\mathcal{F} \end{tikzcd} \quad , \quad \begin{tikzcd} N_k\mathcal{E}' \arrow[r] \arrow[d, "g'"] & N_0\mathcal{E}' \arrow[d, "g'"] \\ N_k\mathcal{F} \arrow[r] & N_0\mathcal{F} \end{tikzcd}$$

are pullback squares. Passage to representing spaces turns them into homotopy cartesian squares by A.6, since the maps $N_0\mathcal{E} \to N_0\mathcal{F}$ and $N_0\mathcal{E}' \to N_0\mathcal{F}$ have the concordance lifting property. Hence it suffices to consider the case $k = 0$,

$$N_0 u\colon N_0\mathcal{E} \longrightarrow N_0\mathcal{E}'.$$

Again, $N_0\mathcal{E} \to N_0\mathcal{F}$ and $N_0\mathcal{E}' \to N_0\mathcal{F}$ have the concordance lifting property and $N_0 u$ induces a weak equivalence (homology equivalence) of the fibers. By Proposition A.6, the fibers turn into homotopy fibers upon passage to representing spaces. Consequently $N_0 u\colon N_0\mathcal{E} \to N_0\mathcal{E}'$ is a homotopy equivalence (homology equivalence). □

4.2. *Armlets.* We begin by defining sheaves $\mathcal{W}^{\mathscr{A}}$ and $h\mathcal{W}^{\mathscr{A}}$ on $\mathscr{X}$ with values in partially ordered sets, and natural transformations

$$\begin{tikzcd} & \mathscr{P}osets \arrow[dd, "\text{forget}"] \\ \mathscr{X} \arrow[ur, bend left, "\mathcal{W}^{\mathscr{A}}"] \arrow[dr, bend right, "\mathcal{W}^0"'] & \Downarrow \\ & \mathscr{S}ets \end{tikzcd} \qquad \begin{tikzcd} & \mathscr{P}osets \arrow[dd, "\text{forget}"] \\ \mathscr{X} \arrow[ur, bend left, "h\mathcal{W}^{\mathscr{A}}"] \arrow[dr, bend right, "h\mathcal{W}^0"'] & \Downarrow \\ & \mathscr{S}ets \end{tikzcd}$$

where $\mathcal{W}^0$ and $h\mathcal{W}^0$ are the sheaves introduced in Section 2.5, weakly equivalent to $\mathcal{W}$ and $h\mathcal{W}$, respectively.

Definition 4.7. An *armlet* for an element $(\pi, f) \in \mathcal{W}^0(X)$ is a compact interval $A \subset \mathbb{R}$ such that $0 \in \mathrm{int}(A)$ and f is fiberwise transverse to the endpoints of A.

Definition 4.8. An *armlet* for an element $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$ is a compact interval $A \subset \mathbb{R}$ such that $0 \in \mathrm{int}(A)$ and

(i) f is fiberwise transverse to the endpoints of A;

(ii) $\hat{f}$ is integrable on an open neighborhood of $f^{-1}(\mathbb{R} \smallsetminus \mathrm{int}(A))$.

We introduce a partial ordering on elements of $\mathcal{W}^0(X)$ or $h\mathcal{W}^0(X)$ equipped with armlets, namely for elements of $\mathcal{W}^0(X)$:

$$(\pi, f, A) \le (\pi', f', A') \quad \text{if} \quad (\pi, f) = (\pi', f') \quad \text{and} \quad A \subset A'$$

and similarly for elements of $h\mathcal{W}^0(X)$.

Definition 4.9. For a connected X in $\mathscr{X}$ we let $\mathcal{W}^{\mathscr{A}}(X)$ denote the partially ordered set of elements (π, f, A) with A an armlet for $(\pi, f) \in \mathcal{W}^0(X)$. Similarly, $h\mathcal{W}^{\mathscr{A}}(X)$ is the partially ordered set of elements $(\pi, \hat{f}, A)$ where $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$ and A is an armlet for $(\pi, \hat{f})$. If X is not connected we (must) define

$$\mathcal{W}^{\mathscr{A}}(X) = \textstyle\prod_i \mathcal{W}^{\mathscr{A}}(X_i)\,, \qquad h\mathcal{W}^{\mathscr{A}}(X) = \textstyle\prod_i h\mathcal{W}^{\mathscr{A}}(X_i)$$

where the X_i are the path components of X.

Any sheaf $\mathcal{F}\colon \mathscr{X} \to \mathscr{S}ets$ can be considered to be a sheaf with a trivial category structure, so that each $\mathcal{F}(X)$ is the object set of a category which has only identity morphisms. In this case an element $(\mathscr{Y}, \varphi_{\bullet\bullet})$ of $\beta\mathcal{F}(X)$ reduces to a pair consisting of a locally finite open covering of X, indexed by J, and a single element $\varphi \in \mathcal{F}(X)$, namely, the unique element restricting to $\varphi_{SS} \in \mathcal{F}(Y_S)$ for every finite nonempty subset S of J. Thus $\beta\mathcal{F} \cong \beta\star \times \mathcal{F}$ where $\star$ denotes the terminal sheaf, again viewed as a sheaf with category values. In particular there is a forgetful projection $\beta\mathcal{F} \longrightarrow \mathcal{F}$ which is a weak equivalence, since $|\beta\star|$ is contractible by Theorem 4.2.

Proposition 4.10. *The forgetful maps $\beta\mathcal{W}^{\mathscr{A}} \to \mathcal{W}^0$ and $\beta h\mathcal{W}^{\mathscr{A}} \to h\mathcal{W}^0$ are weak equivalences of sheaves.*

The proof of Proposition 4.10 will be broken up into the proofs of the following three lemmas.

Lemma 4.11. *Let X be in $\mathscr{X}$ and $(\pi, f) \in \mathcal{W}^0(X)$. Every $x \in X$ has an open neighborhood U in X such that the image of (π, f) in $\mathcal{W}^0(U)$ admits an armlet.*

Proof. Write $\pi : E \to X$ and $E_x = \pi^{-1}(x)$. By Sard's theorem, we can find numbers $a < 0$ and $b > 0$ such that $f_x : E_x \to \mathbb{R}$ is transverse to a and b (in other words, a and b are regular values of f_x). Let $A = [a, b]$. Let $C \subset E$ be the closed subset consisting of all $z \in E$ where f has a fiberwise singularity and $f(z) = a$ or $f(z) = b$. Then $\pi|C$ is proper and so $\pi(C)$ is a closed subset of X. Let $U = X \smallsetminus \pi(C)$. $\square$

Lemma 4.12. *With the assumptions of Lemma* 4.11, *there exists an element of $\beta\mathcal{W}^{\mathscr{A}}(X)$ which, under the forgetful transformation $\beta\mathcal{W}^{\mathscr{A}} \to \mathcal{W}^0$, maps to (π, f).*

Proof. Choose a locally finite covering of X by open subsets Y_j, where $j \in J$, such that the restriction of (π, f) to each Y_j admits an armlet $A_j \subset \mathbb{R}$. For a finite nonempty subset $S \subset J$ with nonempty Y_S let $A_S = \bigcap_{j \in S} A_j$. Then A_S is an armlet for the restriction of (π, f) to Y_S. Therefore, given nonempty finite $R, S \subset J$ with $R \subset S$ and $Y_S \neq \emptyset$, we can define $\varphi_{RS} \in N_1\mathcal{W}^{\mathscr{A}}(Y_S)$ to be the relation

$$(\pi, f, A_S)|Y_S \leq (\pi, f, A_R)|Y_S.$$

The data φ_{RS} then constitute an element of $\beta\mathcal{W}^{\mathscr{A}}(X)$ which clearly projects to $(\pi, f) \in \mathcal{W}^0(X)$. $\square$

It follows from the two previous lemmas that the forgetful map from $\beta\mathcal{W}^{\mathscr{A}}[X]$ to $\mathcal{W}^0[X]$ is surjective for any X in $\mathscr{X}$. What we really need in order to prove the first half of Proposition 4.10 is the relative surjectivity as in Proposition 2.18. This comes from the next lemma, in which we assume that our fixed indexing set J is *uncountable*. (The assumption is not needed in Proposition 4.10 because the homotopy type of $|\beta...|$ is independent of the cardinality of J as long as J is infinite.)

Lemma 4.13. *For X in $\mathscr{X}$, let $(\pi, f) \in \mathcal{W}^0(X)$. Let C be a closed subset of X and suppose that a germ of lifts of (π, f) across $\beta\mathcal{W}^{\mathscr{A}} \longrightarrow \mathcal{W}^0$ has been specified near C. Then there exists an element in $\beta\mathcal{W}^{\mathscr{A}}(X)$ which lifts $(\pi, f) \in \mathcal{W}(X)$ and extends the prescribed germ of lifts near C.*

Proof. Let U be a sufficiently small open neighborhood of C in X so that the prescribed germ of lifts is represented by an actual lift of $(\pi, f)|U$ across $\beta\mathcal{W}^{\mathscr{A}}(U) \longrightarrow \mathcal{W}^0(U)$. This gives us a locally finite covering $\mathscr{Y}'$ of U, and for each nonempty finite $S \subset J$ and each $z \in \pi_0(Y'_S)$, a compact interval $A'_{S,z} \subset \mathbb{R}$ such that $0 \in \operatorname{int}(A'_{S,z})$. We have $A'_{S,z} \subset A'_{R,\bar{z}}$ if $R \subset S$ and $\bar{z}$ is the image of z under $\pi_0(Y'_S) \to \pi_0(Y'_R)$. Making U smaller if necessary, we can assume that the covering $\mathscr{Y}'$ is locally finite in the strong sense that every $x \in X$ has a neighborhood in X which intersects only finitely many of the Y'_j.

Now we make a locally finite covering of X by open subsets Y_j as follows. For $j \in J$ such that Y'_j is nonempty, let $Y_j = Y'_j$. For all other $j \in J$ (and there are many such since J is uncountable) define Y_j in such a way that Y_j avoids a fixed neighborhood of C and the restriction of (π, f) to each path component $z \in \pi_0(Y_j)$ admits an armlet $A_{j,z}$.

It remains to find enough armlets. We need one armlet $A_{S,z} \subset \mathbb{R}$ for each nonempty finite $S \subset J$ and every component $z \in \pi_0(Y_S)$. These armlets must satisfy $A_{S,z} \subset A_{R,\bar{z}}$ if $R \subset S$ and $\bar{z}$ is the image of z under $\pi_0(Y_S) \to \pi_0(Y_R)$. But, reasoning as in the proof of Lemma 4.12, we find that it is enough to say what $A_{S,z} = A_{j,z}$ should be when S is a singleton $\{j\}$. We have already said it in the cases where $Y_j \neq Y'_j$; in the other cases we say $A_{j,z} := A'_{j,z}$. □

The proof of the second half of Proposition 4.10 goes like the proof of the first half, except for one additional observation which is related to condition (ii) in Definition 4.8. For X in $\mathscr{X}$ let $h_c\mathcal{W}^0(X)$ consist of all $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$, with $\pi \colon E \to X$ etc., such that $\hat{f}$ is integrable on some open $U \subset E$ and π restricted to $E \smallsetminus U$ is proper.

LEMMA 4.14. *The inclusion of sheaves $h_c\mathcal{W}^0 \hookrightarrow h\mathcal{W}^0$ is a weak equivalence.*

Proof. Let $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$, with $\pi \colon E \to X$. Choose an open $U \subset E$ such that π restricted to $E \smallsetminus U$ is proper and such that the closure of U has empty intersection with $f^{-1}(0)$. Using the convexity of the fibers of $J^2_\pi(E, \mathbb{R}) \to E$, especially over points $z \in U$, one may deform $\hat{f}$ (leaving f unchanged) in such a way that it becomes integrable on U. This shows that $h_c\mathcal{W}^0[X] \to h\mathcal{W}^0[X]$ is surjective. The argument can easily be refined to prove a relative statement as in the hypothesis of Proposition 2.18. □

4.3. *Proof of Theorem* 1.2. According to Lemma 2.23 and Proposition 4.10 it remains to show that

$$j^2_\pi \colon \beta\mathcal{W}^{\mathscr{A}} \to \beta h\mathcal{W}^{\mathscr{A}}$$

is a weak equivalence. To this end we introduce a new sheaf

$$\mathcal{T}^{\mathscr{A}} \colon \mathscr{X} \longrightarrow \mathscr{P}osets.$$

Suppose given a smooth submersion $\pi \colon E \to X$ with $(d+1)$-dimensional fibers and a Θ-orientation on $T^\pi E$, as in Definitions 2.6 and 2.7. We consider pairs (ψ, A) where $\psi \colon E \to \mathbb{R}$ is a smooth function such that $(\pi, \psi) \colon E \to X \times \mathbb{R}$ is proper, $A \subset \mathbb{R}$ is a compact interval with $0 \in \mathrm{int}(A)$, and ψ is fiberwise transverse to ∂A. There is no restriction on the fiberwise singularities that ψ might have.

Definition 4.15. For a connected X in $\mathscr{X}$, the set $\mathcal{T}^{\mathscr{A}}(X)$ consists of triples (π, ψ, A) as above, modulo the equivalence relation which has (π, ψ, A) equivalent to (π, ζ, A) if $\psi^{-1}(A) = \zeta^{-1}(A)$ and the support of $\psi - \zeta$ is contained in the interior of $\psi^{-1}(A)$.

As for $\mathcal{W}^{\mathscr{A}}$, we get $\mathcal{T}^{\mathscr{A}} : \mathscr{X} \to \mathscr{P}osets$. Moreover there is an obvious commutative diagram of sheaves

(4.3)
$$\begin{tikzcd} \mathcal{W}^{\mathscr{A}} \arrow[rr, "j_\pi^2"] \arrow[dr, "p"'] & & h\mathcal{W}^{\mathscr{A}} \arrow[dl, "q"] \\ & \mathcal{T}^{\mathscr{A}} & \end{tikzcd}$$

where $p(\pi, f, A)$ and $q(\pi, \hat{f}, A)$ are the equivalence classes of (π, f, A); in the second case f is the underlying function of $\hat{f}$.

Let (π, ψ, A) be a representative of an element of $\mathcal{T}^{\mathscr{A}}(X)$ with $\pi : E \to X$, $\psi : E \to \mathbb{R}$ and $A \subset \mathbb{R}$. The manifold $\psi^{-1}(A)$ is independent of the choice of representative for the equivalence class, and $\pi|\psi^{-1}(A)$ is a proper submersion; hence a smooth fiber bundle by Ehresmann's fibration Lemma [4]. Moreover, near the boundary $\partial\psi^{-1}(A) = \psi^{-1}(\partial A)$, the function ψ is independent of the choice of representative.

LEMMA 4.16. *The maps p and q in* (4.3) *have the concordance lifting property.*

Proof. We only give the proof for p, since the proof for q is much the same. Suppose given a concordance $[\pi, \psi, A] \in \mathcal{T}^{\mathscr{A}}(X \times \mathbb{R})$ and a lift to $\mathcal{W}^{\mathscr{A}}(X \times 0)$ of its restriction to $X \times 0$. The projection

$$\psi^{-1}(A) \xrightarrow{\pi} X \times \mathbb{R} \tag{4.4}$$

is a smooth manifold bundle. Hence there exists a diffeomorphism $N \times \mathbb{R} \cong \psi^{-1}(A)$ over $X \times \mathbb{R}$, where $N = \psi^{-1}(A) \cap \pi^{-1}(X \times 0)$. But what we need here is a diffeomorphism

$$u : N \times \mathbb{R} \longrightarrow \psi^{-1}(A)$$

over $X \times \mathbb{R}$ such that $\psi(u(z,t)) = \psi(u(z,0))$ for all (z,t) near $\partial N \times \mathbb{R}$, and of course $u(z,0) = z$ for all $z \in N$. Constructing such a diffeomorphism u is equivalent to constructing a smooth vector field $\xi = du/dt$ on $\psi^{-1}(A)$ which

(i) covers the vector field $(x,t) \mapsto (0,1) \in TX_x \times T\mathbb{R}_t$ on $X \times I$,

(ii) satisfies $\langle d\psi, \xi \rangle \equiv 0$ near $\psi^{-1}(\partial A)$.

(Actually ξ is also prescribed on a neighborhood of $\psi^{-1}(X \times C)$ where $C = \mathbb{R} \smallsetminus]0,1[$, due to the details in Definition 2.3.) This problem has local solutions which can be pieced together by means of a partition of unity on $\psi^{-1}(A)$. Hence u with the required properties exists.

Now we define the lifted concordance $(\pi, f, A) \in \mathcal{W}^{\mathscr{A}}(X \times I)$ in such a way that $f(u(z,t)) = f(u(z,0))$ for $(z,t) \in N \times \mathbb{R}$, bearing in mind that $f(u(z,0)) = f(z)$ is prescribed for all $z \in N$ and f must equal ψ outside $u(N \times I) = \psi^{-1}(A)$. $\square$

Proposition 4.17. *The fiberwise jet prolongation map*

$$j^2_\pi : |\beta\mathcal{W}^{\mathscr{A}}| \longrightarrow |\beta h\mathcal{W}^{\mathscr{A}}|$$

induces an isomorphism on integral homology.

Proof. This will be deduced from Proposition 4.6 and diagram (4.3). Both maps p and q in (4.3) are transport projections in the sense of 4.4. We must determine the fibers of p and q and check that j^2_π induces a homology equivalence between fibers over the same point.

We first determine the fiber $p^{-1}(\tau)$ of

$$p : \mathcal{W}^{\mathscr{A}} \longrightarrow \mathcal{T}^{\mathscr{A}}$$

over an element $\tau = [F, \psi, A] \in \mathcal{T}^{\mathscr{A}}(\star)$. That is, for each X in $\mathscr{X}$ we are interested in the subset of $\mathcal{W}^{\mathscr{A}}(X)$ which maps to the element $[\pi, \psi \circ \mathrm{pr}_F, A]$ of $\mathcal{T}^{\mathscr{A}}(X)$, where π and pr_F are the projections $F \times X \to X$ and $F \times X \to F$, respectively. This subset consists of $(\pi, f, A) \in \mathcal{W}^{\mathscr{A}}(X)$ with π and A as above, where $f : F \times X \to \mathbb{R}$ satisfies the conditions

(i) $\mathrm{supp}(f - \psi \circ \mathrm{pr}_F) \subset \mathrm{int}(\psi^{-1}(A)) \times X$,

(ii) $f(\psi^{-1}(A) \times X) \subset A$.

Because of (i), we can identify the fiber of p over τ with a subsheaf of the sheaf taking X in $\mathscr{X}$ to the set of smooth maps from X to

$$\Phi(\psi^{-1}(A), \mathfrak{A}, \psi),$$

using the notation of (4.1). Similarly, the fiber $q^{-1}(\tau)$ of q in (4.3) over the same element $\tau \in \mathcal{T}^{\mathscr{A}}(\star)$ can be identified with a subsheaf of the sheaf taking X in $\mathscr{X}$ to the set of smooth maps from X to

$$h\Phi(\psi^{-1}(A), \mathfrak{A}, \psi).$$

The inclusions of these subsheaves are weak equivalences by inspection. (That is to say, condition (ii) means nothing after passage to concordance classes.) Thus the representing spaces $|p^{-1}(\tau)|$ and $|q^{-1}(\tau)|$ have canonical comparison maps to $\Phi(\psi^{-1}(A), \mathfrak{A}, \psi)$ and $h\Phi(\psi^{-1}(A), \mathfrak{A}, \psi)$, respectively, which are homotopy equivalences. With these as identifications, the jet prolongation map from $|p^{-1}(\tau)|$ to $|q^{-1}(\tau)|$ turns into a special case of (4.1), and so is a homology equivalence by Vassiliev's first main theorem. $\square$

Combining Lemma 2.23, Proposition 4.10 and Proposition 4.17, we get that

$$j^2_\pi : |\mathcal{W}| \longrightarrow |h\mathcal{W}|$$

induces an isomorphism in homology. Both $|\mathcal{W}|$ and $|h\mathcal{W}|$ are spaces with a monoid structure up to homotopy (cf. the proof of Theorem 3.8) and j^2_π respects this additional structure. The target $|h\mathcal{W}|$ is an infinite loop space by Theorem 3.8, hence it is group complete. (That is, the monoid $\pi_0|h\mathcal{W}|$ is a group.) Since $H_*(j^2_\pi;\mathbb{Z})$ is an isomorphism, especially when $*=0$, the source $|\mathcal{W}|$ is also group complete. It is well known that the connected components of a space with a group complete monoid structure up to homotopy are *simple*, and that a map between simple spaces is a homology equivalence if and only if it is a homotopy equivalence. This completes the proof of Theorem 1.2. □

5. Some homotopy colimit decompositions

The organization and the main results of this section can be summarized in a commutative diagram of sheaves on $\mathscr{X}$ and maps of sheaves

(5.1)
$$\begin{tikzcd}
\mathcal{W} \arrow[r] & \mathcal{W}_{\mathrm{loc}} \\
\mathcal{W}^\mu \arrow[u, "\simeq"'] \arrow[r] & \mathcal{W}^\mu_{\mathrm{loc}} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\mathcal{L} \arrow[u, "\simeq"'] \arrow[r] & \mathcal{L}_{\mathrm{loc}} \\
\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_T \arrow[u, "\simeq"'] \arrow[r] \arrow[d, "\simeq"] & \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_{\mathrm{loc},T} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{W}_T \arrow[r] & \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{W}_{\mathrm{loc},T}\,.
\end{tikzcd}$$

The symbol $\simeq$ indicates weak equivalences. The homotopy colimits in the diagram are homotopy colimits in the category of sheaves on $\mathscr{X}$, as in Definition 4.3. But their representing spaces can be regarded as homotopy colimits in the category of spaces according to Lemma B.9. The top row of diagram (5.1) is the inclusion map $\mathcal{W} \to \mathcal{W}_{\mathrm{loc}}$. The bottom row is what we eventually want to substitute for the top row in order to prove Theorem 1.5.

The following preliminary remarks about (5.1) might help the reader through this rather demanding section.

The elements of $\mathcal{W}(X)$ and $\mathcal{W}_{\mathrm{loc}}(X)$ are families, parametrized by X, of $(d+1)$-manifolds E_x equipped with, among other things, Morse functions $f_x\colon E_x \to \mathbb{R}$. The same description applies to $\mathcal{W}^\mu(X)$, $\mathcal{W}^\mu_{\mathrm{loc}}(X)$, $\mathcal{L}(X)$ and $\mathcal{L}_{\mathrm{loc}}(X)$ in the second and third row of (5.1), except that we ask for more structure around the critical points. In particular, in the important case of $\mathcal{L}(X)$ we insist on proper Morse functions f_x whose critical points $z \in E_x$ are separately enclosed in certain standard neighborhoods. Each of these standard neighborhoods $N_z \subset E_x$ is a $(d+1)$-manifold with boundary; the restricted map $f_x|N_z$ is proper, regular on ∂N_z and has no critical points in the interior of N_z except z. These data will enable us later on to move the critical values of f up or down, independently of each other.

In going from the third row of (5.1) to the fourth row, we are adding "local" decisions which, for each critical point in sight, specify whether the corresponding critical value should eventually be moved towards $-\infty$, $+\infty$ or 0. For more precision, suppose that we are dealing with a family $(\pi, f)\colon E \to X \times \mathbb{R}$ of $(d+1)$-manifolds and proper Morse functions, plus standard neighborhoods for the critical points, i.e., an element of $\mathcal{L}(X)$. Let $\Sigma(\pi, f)$ be the fiberwise singularity set. Recall that the projection $\Sigma(\pi, f) \to X$ is étale. On some connected components (alias sheets) of $\Sigma(\pi, f)$, the map f might neither be bounded above nor below. This makes a reasonable partition of $\Sigma(\pi, f)$ into a positive, a negative and a neutral part *globally* impossible. But the problem can be solved *locally* in X. Namely, for any $x \in X$ there exist an open neighborhood U_x of x in X, and a partition of $\Sigma(\pi, f) \cap \pi^{-1}(U_x)$ into three closed parts: a "positive" part where f is bounded below, a "negative" part where f is bounded above, and a "neutral" part where f is bounded below and above. (The partition is usually not unique.) The neutral part will always be a finite covering space of U_x and, by making U_x smaller, we can assume that it is trivialized, i.e. identified with $T \times U_x$ for a finite set T with some extra structure. By fixing T and adding these trivialization and partition data to the definition of $\mathcal{L}(X)$ or $\mathcal{L}_{\mathrm{loc}}(X)$, we obtain the definitions of $\mathcal{L}_T(X)$ and $\mathcal{L}_{\mathrm{loc},T}(X)$. The local existence statement just described translates into homotopy colimit decompositions, i.e., the equivalence between the third and fourth rows of (5.1). This should not come as a surprise, since our definition of the sheaf-theoretic homotopy colimit, Definition 4.3, involves open coverings and therefore obviously has a "local" flavor.

Finally to pass from the fourth row in (5.1) to the fifth, we produce concordances which remove critical point sheets labelled positive or negative and which move the remaining critical values towards 0. By considering a regular level, we are led to weak equivalences $\mathcal{L}_T \simeq \mathcal{W}_T$ and $\mathcal{L}_{\mathrm{loc},T} \simeq \mathcal{W}_{\mathrm{loc},T}$ where $\mathcal{W}_T(X)$ and $\mathcal{W}_{\mathrm{loc},T}(X)$ are defined in terms of bundles of closed d-manifolds on X and fiberwise surgery data.

5.1. *Description of main results.* We now give a description of the lower row in diagram (5.1). This begins with a definition of the category $\mathscr{K}$ by which the homotopy colimits are indexed.

Definition 5.1. An object of $\mathscr{K}$ is a finite set S equipped with a map to the set $\{0, 1, 2, \ldots, d+1\}$. A morphism from S to T is a pair (k, ε) where k is an injective map, over $\{0, 1, \ldots, d+1\}$, from S to T and ε is a function

$T \smallsetminus k(S) \to \{-1,+1\}$. The composition of two morphisms $(k_1, \varepsilon_1)\colon S \to T$ and $(k_2, \varepsilon_2)\colon T \to U$ is $(k_2k_1, \varepsilon_3)\colon S \to U$ where ε_3 agrees with ε_2 outside $k_2(T)$ and with $\varepsilon_1 \circ {k_2}^{-1}$ on $k_2(T \smallsetminus k_1(S))$.

Many times below we encounter riemannian vector bundles $\omega\colon V \to Y$ equipped with a fiberwise isometry $\varrho\colon V \to V$ over Y such that $\varrho^2 = \mathrm{id}\colon V \to V$. We call a vector bundle with this additional structure a *Morse vector bundle*, and if $Y = \star$, a *Morse vector space*.

Definition 5.2. Let T be an object of $\mathscr{K}$. For X in $\mathscr{X}$, let $\mathcal{W}_{\mathrm{loc},T}(X)$ be the set of smooth, $(d+1)$-dimensional Morse vector bundles $\omega\colon V \longrightarrow T \times X$ equipped with a Θ-orientation and subject to the following conditions.

(i) For $(t,x) \in T \times X$, the dimension of the fixed point space of $-\varrho$ acting on the fiber $V_{(t,x)}$ is equal to the label of t in $\{0,1,\ldots,d+1\}$;

(ii) The composition $V \to T \times X \to X$ is a graphic map.

A smooth map $g\colon X \to Y$ induces a map $\mathcal{W}_{\mathrm{loc},T}(Y) \to \mathcal{W}_{\mathrm{loc},T}(X)$, given by pullback of vector bundles V on $T \times Y$ along $\mathrm{id} \times g\colon T \times X \to T \times Y$. (The underlying set should be the graphic pullback of $V \to Y$ along g.) This makes $\mathcal{W}_{\mathrm{loc},T}$ into a sheaf on $\mathscr{X}$.

In Definition 5.2, the involution on V leads to an orthogonal vector bundle splitting $V = V^{\varrho} \oplus V^{-\varrho}$, where V^{ϱ} consists of the vectors fixed by ϱ and $V^{-\varrho}$ consists of the vectors fixed by $-\varrho$. We write $D(V^{\varrho})$ and $S(V^{-\varrho})$ for the disk and sphere bundles associated with V^{ϱ} and $V^{-\varrho}$, respectively. The vertical tangent bundle of the projection

$$D(V^{\varrho}) \times_{T\times X} S(V^{-\varrho}) \quad \longrightarrow \quad X$$

inherits a preferred Θ-orientation from V, described in detail at the end of Section 5.5.

Definition 5.3. For T in $\mathscr{K}$, a sheaf $\mathcal{W}_T$ on $\mathscr{X}$ is defined as follows. For X in $\mathscr{X}$, an element of $\mathcal{W}_T(X)$ consists of

(i) a smooth graphic bundle $q\colon M \to X$ of closed d-manifolds, with a Θ-orientation of its fiberwise tangent bundle;

(ii) an element (V, ϱ) of $\mathcal{W}_{\mathrm{loc},T}(X)$;

(iii) a smooth embedding over X respecting the fiberwise tangential Θ-orientations,

$$e\colon D(V^{\varrho}) \times_{T\times X} S(V^{-\varrho}) \quad \longrightarrow \quad M\,.$$

The sheaves in Definitions 5.2 and 5.3 depend contravariantly on the variable T in $\mathscr{K}$. This is clear in the case of Definition 5.2: A morphism $(k,\varepsilon)\colon S\to T$ in $\mathscr{K}$ induces a map from $\mathcal{W}_{\mathrm{loc},T}(X)$ to $\mathcal{W}_{\mathrm{loc},S}(X)$ given by pullback of vector bundles along the map $k\times\mathrm{id}$ from $S\times X$ to $T\times X$. (More precisely, for $(V,\varrho)\in\mathcal{W}_{\mathrm{loc},T}(X)$ with bundle projection $\omega\colon V\to T\times X$, we let $(k,\varepsilon)^*(V,\varrho)=(V',\varrho')$ where $V'=\omega^{-1}(k(S)\times X)$ with bundle projection $\omega'=\omega\circ(k\times\mathrm{id})^{-1}$.)

The case of Definition 5.3 is much more interesting. Let $(k,\varepsilon)\colon S\to T$ be a morphism in $\mathscr{K}$. If k is bijective, there is an obvious identification $\mathcal{W}_T\cong\mathcal{W}_S$ and this is the induced map. Therefore we may assume that k is an inclusion $S\hookrightarrow T$. Then we can reduce to the case where $T\smallsetminus S$ has exactly one element, a. This case has two subcases: $\varepsilon(a)=+1$ and $\varepsilon(a)=-1$.

Definition 5.4. Let $(k,\varepsilon)\colon S\to T$ be a morphism in $\mathscr{K}$ where k is an inclusion and $T\smallsetminus S=\{a\}$ with $\varepsilon(a)=+1$. We describe the induced map

$$\mathcal{W}_T(X)\longrightarrow\mathcal{W}_S(X).$$

Let (q,V,ϱ,e) be an element of $\mathcal{W}_T(X)$, with $q\colon M\to X$. Map this to an element of $\mathcal{W}_S(X)$ by keeping $q\colon M\to X$, restricting V to $S\times X$ and restricting ϱ and e accordingly.

Definition 5.5. Let $(k,\varepsilon)\colon S\to T$ be a morphism in $\mathscr{K}$ where k is an inclusion and $T\smallsetminus S=\{a\}$ with $\varepsilon(a)=-1$. For X in $\mathscr{X}$, the induced map

$$\mathcal{W}_T(X)\longrightarrow\mathcal{W}_S(X)$$

is defined as follows. Let (q,V,ϱ,e) be an element of $\mathcal{W}_T(X)$, with $q\colon M\to X$. Map this to the element (q',V',ϱ',e') of $\mathcal{W}_S(X)$ where

(i) $q'\colon M'\to X$ is obtained from $q\colon M\to X$ by fiberwise surgery on the embedded bundle of thickened spheres $e\big(D(V^{\varrho}|X_a)\times_{X_a}S(V^{-\varrho}|X_a)\big)$, where X_a means $a\times X$;

(ii) (V',ϱ') is the restriction of (V,ϱ) to $S\times X$;

(iii) e' is obtained from e by restriction.

Remark 5.6. For now, the main point is that the fiberwise surgery in (i) amounts to removing the interior of the embedded thickened sphere bundle and gluing in a copy of $D(V^{-\varrho}|X_a)\times_{X_a}S(V^{\varrho}|X_a)$ instead. More details will be given later, at the end of Section 5.5. Note that when $V^{-\varrho}=0$, the embedded thickened sphere bundle whose interior we have to remove is empty. In this case the fiberwise surgery consists in adding a (disjoint) copy of the sphere bundle $S(V)|X_a$ to M. If $V^{\varrho}=0$, the fiberwise surgery removes a (disjoint) copy of $S(V)|X_a$.

There is a forgetful map of sheaves $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$. It has the concordance lifting property, so that by Corollary A.8, the representing spaces of its fibers are the homotopy fibers of the induced map of representing spaces

$$|\mathcal{W}_T| \to |\mathcal{W}_{\mathrm{loc},T}|.$$

It is easy to see that the representing space of any fiber of $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ is a classifying space for certain bundles of compact Θ-oriented d-manifolds with a prescribed boundary; cf. Section 5.6.

5.2. *Morse singularities, Hessians and surgeries.* We begin by recalling some well known facts about elementary and multi-elementary Morse functions. The reader is referred to [28, Ch. I] and [29] for more details in the nonparametrized situation. By an *elementary* Morse function we shall mean a proper smooth map $E \to \mathbb{R}$ which is regular on ∂E and has exactly one critical point in $E \smallsetminus \partial E$ which is nondegenerate. By a *multi-elementary* Morse function we mean a proper smooth map $E \to \mathbb{R}$ which is regular on ∂E and has finitely many critical points in $E \smallsetminus \partial E$, all nondegenerate and all with the same critical value.

Let $V = (V, \langle\, ,\rangle, \varrho)$ be a Morse vector space. The function

$$f_V\colon V \to \mathbb{R}\ , \qquad f_V(v) = \langle v, \varrho v\rangle \tag{5.2}$$

is a Morse function on V with exactly one critical point. If we write $V = V^{\varrho} \oplus V^{-\varrho}$, then the fomula for f_V becomes

$$f_V(v) = \|v_+\|^2 - \|v_-\|^2$$

where v_+ and v_- are the components of v in V^{ϱ} and $V^{-\varrho}$, respectively. The gradient of f_V is everywhere perpendicular to the gradient of $v \mapsto \|v_+\|^2\|v_-\|^2$, so that the latter function is constant on the trajectories of the gradient flow of f_V. This motivates the following definition.

Definition 5.7. $\mathrm{sdl}(V,\varrho) = \{v \in V \mid \|v_+\|^2\|v_-\|^2 \le 1\}$.

If $V^{\varrho} = 0$ or $V^{-\varrho} = 0$, then $\mathrm{sdl}(V,\varrho) = V$. For arbitrary V and ϱ, the formula

$$v \mapsto (\|v_-\|v_+,\ \|v_-\|^{-1}v_-,\ f_V(v)) \tag{5.3}$$

defines a smooth embedding of $\mathrm{sdl}(V,\varrho) \smallsetminus V^{\varrho}$ in $D(V^{\varrho}) \times S(V^{-\varrho}) \times \mathbb{R}$, with complement $0 \times S(V^{-\varrho}) \times [0,\infty[$. It respects boundaries and is a map over $\mathbb{R}$, where we use the restriction of f_V on the source and the function $(x,y,t) \mapsto t$ on the target.

Dually, the formula

$$v \mapsto (\|v_+\|v_-,\ \|v_+\|^{-1}v_+,\ f_V(v)) \tag{5.4}$$

defines a smooth embedding of $\mathrm{sdl}(V,\varrho)\smallsetminus V^{-\varrho}$ in $D(V^{-\varrho})\times S(V^{\varrho})\times\mathbb{R}$, with complement $0\times S(V^{\varrho})\times\,]-\infty,0]$. It respects boundaries and is a map over $\mathbb{R}$.

The map f_V in (5.2) restricted to $\mathrm{sdl}(V,\varrho)$ is a good local model for elementary Morse functions. Let M be any smooth compact manifold and let

$$e\colon D(V^{\varrho})\times S(V^{-\varrho})\to M\smallsetminus\partial M \tag{5.5}$$

be a codimension zero embedding ("surgery data"). Then in $M\times\mathbb{R}$ we have an embedded copy of $D(V^{\varrho})\times S(V^{-\varrho})\times\mathbb{R}$. We can remove its interior and glue in $\mathrm{sdl}(V,\varrho)$ instead, using formula (5.3) to identify the boundary of $\mathrm{sdl}(V,\varrho)$ with the boundary of $D(V^{\varrho})\times S(V^{-\varrho})\times\mathbb{R}$. The result is a smooth manifold $\mathrm{Trc}(e)$ of dimension $\dim(M)+1$. For example, if $M=S^1$ and $\dim(V^{\varrho})=1$, the cylinder $M\times\mathbb{R}$ is replaced by an (infinite) pair of pants.

Definition 5.8. The *long trace* of e, denoted $\mathrm{Trc}(e)$, is the pushout of the two smooth codimension zero embeddings

$$\begin{aligned}&\mathrm{sdl}(V,\varrho)\smallsetminus V^{\varrho}\xrightarrow{(e\times\mathrm{id})\circ(5.3)}(M\times\mathbb{R})\smallsetminus e(0\times S(V^{-\varrho}))\times[0,\infty[\,,\\ &\mathrm{sdl}(V,\varrho)\smallsetminus V^{\varrho}\hookrightarrow\mathrm{sdl}(V,\varrho).\end{aligned}\tag{5.6}$$

For example, if $V^{-\varrho}=0$, then $\mathrm{sdl}(V,\varrho)=V$ and $\mathrm{sdl}(V,\varrho)\smallsetminus V^{\varrho}$ is empty, so that $\mathrm{Trc}(e)$ becomes the disjoint union of $M\times\mathbb{R}$ and $V=V^{\varrho}$. Note that M can be empty in this case. If $V^{\varrho}=0$, then M contains a codimension zero copy of $S(V)$. The long trace is obtained by removing $S(V)\times[0,\infty[$ from the copy of $S(V)\times\mathbb{R}$ in $M\times\mathbb{R}$ and adding a single point instead, so that $\mathrm{Trc}(e)$ becomes the disjoint union of $(M\smallsetminus\mathrm{im}(e))\times\mathbb{R}$ and $V=V^{-\varrho}$.

Definition 5.8 determines a structure of smooth manifold on $\mathrm{Trc}(e)$ and shows that $\mathrm{Trc}(e)$ comes with a (smooth) elementary Morse function, the *height function*, which is the projection to $\mathbb{R}$ on the complement of the saddle and equal to $v\mapsto\langle v,\varrho v\rangle$ on the glued-in copy of $\mathrm{sdl}(V,\varrho)$. The unique critical point is the origin of $V^{\varrho}\subset\mathrm{Trc}(e)$. The corresponding critical value is 0.

Roughly speaking, every elementary Morse function can be identified with the height function on $\mathrm{Trc}(e)$ for some M and e. This will be illustrated in Section 5.4.

The long trace construction has some obvious generalizations. For example, we can allow simultaneous surgeries on a finite number of pairwise disjoint thickened spheres. In this case the surgery data consist of a finite set T, a Morse vector bundle V on T where $\dim(V)=\dim(M)+1$, and a smooth embedding

$$e\colon D(V^{\varrho})\times_T S(V^{\varrho})\longrightarrow M\smallsetminus\partial M\,.$$

Then $\mathrm{Trc}(e)$ is defined as the manifold obtained from $M\times\mathbb{R}$ by deleting the embedded copy of

$$D(V_t^{\varrho})\times S(V_t^{-\varrho})\times\mathbb{R}$$

for each $t \in T$, and substituting $\mathrm{sdl}(V_t, \varrho)$ for it using formula (5.3) to do the gluing. There is a canonical height function on $\mathrm{Trc}(e)$ which is a Morse function with one critical point for each $t \in T$. The only critical value is 0 (if $T \neq \emptyset$).

We shall use a parametrized version of this construction. Let $q\colon M \to X$ be a bundle of smooth compact n-manifolds, let $V \to T \times X$ be a riemannian vector bundle of fiber dimension $n+1$ with isometric involution ϱ, and let

$$e\colon D(V^{\varrho}) \times_{T\times X} S(V^{-\varrho}) \longrightarrow M \smallsetminus \partial M$$

be a smooth embedding over X. We can regard e as a family of embeddings e_x for $x \in X$, each from a disjoint union of finitely many thickened spheres to a fiber M_x of q. The manifolds $\mathrm{Trc}(e_x)$ for $x \in X$ are the fibers of a smooth bundle

$$E = \mathrm{Trc}(e) \longrightarrow X\,. \tag{5.7}$$

It comes equipped with a smooth height function $f\colon \mathrm{Trc}(e) \longrightarrow \mathbb{R}$ which is fiberwise Morse; if $T \neq \emptyset$, then the unique critical value is 0.

So far we have looked at ways to create nondegenerate critical points, starting with a regular function such as a projection $M \times \mathbb{R} \to \mathbb{R}$. For us the opposite process, that of removing or "regularizing" nondegenerate critical points of a Morse function $N \to \mathbb{R}$, will be more important. This corresponds to going through the long trace construction in reverse. In order to carry over the Θ-orientation it is convenient to first observe that the regularized manifold is diffeomorphic to $N \smallsetminus V^{\varrho}$ and hence inherits a Θ-orientation from N. We owe this observation to S. Galatius. Here are the details.

Choose once and for all a diffeomorphism ψ from $\mathbb{R}$ to $]-\infty, 0\,[$ such that $\psi(t) = t$ for $t < -1/2$, and a smooth nondecreasing function $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(x) = x$ for x close to 0 and $\varphi(x) = 1$ for x close to 1. Let

$$\psi_x(t) = \varphi(x)t + (1-\varphi(x))\psi(t)$$

for $x \in [0,1]$. Then $\psi_0 = \psi$ embeds $\mathbb{R}$ in $\mathbb{R}$ with image $]-\infty, 0\,[$, whereas each ψ_x for $x > 0$ is a diffeomorphism $\mathbb{R} \to \mathbb{R}$. We define proper (regular) functions

$$f_V^+\colon \mathrm{sdl}(V,\varrho) \smallsetminus V^{\varrho} \to \mathbb{R}\ , \qquad f_V^-\colon \mathrm{sdl}(V,\varrho) \smallsetminus V^{-\varrho} \to \mathbb{R}$$

by the formulae

$$f_V^+(v) = \psi_x^{-1}(t)\ , \qquad f_V^-(v) = (-\psi_x)^{-1}(-t) \tag{5.8}$$

where $t = f_V(v)$ and $x = \|v_-\|^2\|v_+\|^2$. These functions agree with f_V on open subsets that contain the entire boundary and the sets

$$\{w \in \mathrm{sdl}(V,\varrho) \mid f_V(w) \leq -1\}\ , \qquad \{w \in \mathrm{sdl}(V,\varrho) \mid f_V(w) \geq +1\}\ ,$$

respectively. Using $f_V^{\pm}$ instead of f_V the embeddings (5.3), (5.4) are replaced by the diffeomorphisms

$$(5.9)\qquad \begin{aligned} \sigma_V^+ \colon \mathrm{sdl}(V,\varrho) \smallsetminus V^{\varrho} &\longrightarrow D(V^{\varrho}) \times S(V^{-\varrho}) \times \mathbb{R}, \\ \sigma_V^- \colon \mathrm{sdl}(V,\varrho) \smallsetminus V^{-\varrho} &\longrightarrow D(V^{-\varrho}) \times S(V^{\varrho}) \times \mathbb{R} \end{aligned}$$

given by

$$\sigma_V^+(v) = (\|v_-\|v_+,\, \|v_-\|^{-1}v_-,\, f_V^+(v)), \quad \sigma_V^-(v) = (\|v_+\|v_-,\, \|v_+\|^{-1}v_+,\, f_V^-(v)).$$

Let $f\colon N \to \mathbb{R}$ be an elementary Morse function with unique critical value 0. By the Morse-Palais lemma, we can choose a Morse vector space V and a codimension zero embedding $\lambda\colon \mathrm{sdl}(V,\varrho) \to N \smallsetminus \partial N$ with the property $f\lambda = f_V$. Define

$$N^{\mathrm{rg}} = N \smallsetminus \lambda(V^{\varrho}) \ , \qquad f^{\mathrm{rg}}\colon N^{\mathrm{rg}} \to \mathbb{R},$$

by $f^{\mathrm{rg}}(x) = f(x)$ for $x \notin \mathrm{im}(\lambda)$ and $f^{\mathrm{rg}}(\lambda(w)) = f_V^+(w)$ for $w \in \mathrm{sdl}(V,\varrho) \smallsetminus V^{\varrho}$. The function f^{rg} is smooth, proper and regular. Any Θ-orientation on TN can obviously be restricted to TN^{rg}. Note that the construction applied to $\mathrm{Trc}(e)$ of Definition 5.8 gives back $M \times \mathbb{R}$, up to a canonical diffeomorphism.

We finish this section with a naturality property of $\mathrm{sdl}(V,\varrho)$, used in Proposition 5.28.

Proposition 5.9. *Suppose given a smooth map $e\colon \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $e(a) = b$. Assume $0 < e'(x) \le 1$ for all $x \in \mathbb{R}$. Then there is a smooth embedding $\tau\colon \mathrm{sdl}(V,\varrho) \to \mathrm{sdl}(V,\varrho)$ with $\tau(0) = 0$ and $\tau'(0) = \sqrt{e'(a)} \cdot \mathrm{id}_V$ such that*

$$(f_V + b) \circ \tau = e \circ (f_V + a).$$

(It is not claimed that the embedding τ respects the boundary of $\mathrm{sdl}(V,\varrho)$. The construction works just as well in a parametrized setting.)

Proof. Without loss of generality, $a = b = 0$; otherwise replace e by e_1 where $e_1(x) = e(x+a) - b$, and note that $e_1(0) = 0$ and that $f_V \circ \tau = e_1 \circ f_V$ implies $(f_V + b) \circ \tau = e \circ (f_V + a)$. Assuming $e(0) = 0$ therefore, we have to define τ in such a way that $f_V \circ \tau = e \circ f_V$. We remark that e is an orientation-preserving embedding since $e'(x) > 0$ for all x.

First define $u\colon \mathbb{R} \to \mathbb{R}$ by $u(x) = e(x)/x$ for $x \neq 0$ and $u(0) = e'(0)$. Then u is smooth, as can be seen from

$$e(x) = \int_0^x e'(t)\ dt = x \int_0^1 e'(xs)\ ds\,.$$

We have $0 < u(x) \le 1$ for $x \in \mathbb{R}$ and $e(x) = u(x) \cdot x$. Let

$$\tau(w) = (u(f_V(w)))^{1/2} w$$

for $w \in \mathrm{sdl}(V, \varrho)$. Then $f_V(\tau(w)) = u(f_V(w)) \cdot f_V(w) = e(f_V(w))$, so that $f_V \circ \tau = e \circ f_V$. It remains to show that τ is an embedding. Write $q(w) = (u(f_V(w)))^{1/2}$ so that $\tau(w) = q(w) \cdot w$. The product rule gives

$$\tau'(w)(h) = (q'(w)(h)) \cdot w + q(w) \cdot h$$

for h in the tangent space T_wV. For $w = 0$ and $h \neq 0$ the right-hand side is clearly nonzero. For $w \neq 0$ the right-hand side can only vanish if h is a scalar multiple of w. It is therefore enough to try $h = w$. This gives $\tau'(w)(w)$ on the left-hand side, which is the derivative of $t \mapsto \tau(tw)$ at $1 \in \mathbb{R}$. If this vanishes, then the derivative of

$$t \mapsto f_V(\tau(tw))$$

at $1 \in \mathbb{R}$ also vanishes. But $f_V(\tau(tw)) = e(f_V(tw))$, and since e' is everywhere nonzero, it follows that $f'_V(w)(w) = 0$ by the chain rule. Since f_V is a quadratic form, this forces $f_V(w) = f_V(tw) = 0$. But then $\tau(tw) = (u(0))^{1/2}tw$ which, as a function of t, certainly has a nonzero derivative at $1 \in \mathbb{R}$, contradiction. Hence $\tau'(w)$ is invertible for every w. Since τ also maps each line segment through $0 \in V$ to itself, it follows immediately that τ is an embedding. $\square$

5.3. *Right-hand column.* Our most important examples of Morse vector bundles are as follows. Let (π, f) be an element of $\mathcal{W}_{\mathrm{loc}}(X)$, cf. Definition 2.9, with $\pi\colon E \to X$. The restriction of the vertical tangent bundle $T^\pi E$ to the fiberwise singularity set $\Sigma = \Sigma(\pi, f)$ comes with an everywhere nondegenerate symmetric bilinear form $\frac{1}{2}H$, where H is the vertical Hessian of f, that is, the second derivative in the fiber direction. See [28, I, §2]. We can choose an orthogonal direct sum decomposition of $T^\pi E|\Sigma$ into a positive definite subbundle and a negative definite subbundle. (The choice is usually not unique, but the space of all such choices is contractible.) By changing the sign of $\frac{1}{2}H$ on the negative definite subbundle, we make $T^\pi E|\Sigma$ into a Morse vector bundle, with an isometric involution which is $-\mathrm{id}$ on the preferred negative definite summand and $+\mathrm{id}$ on the positive definite summand. Note in addition that $\pi|\Sigma$ is an étale map $\Sigma \to X$ and that the restriction of (π, f) to Σ is a proper map from Σ to $X \times \mathbb{R}$.

Definition 5.10. Let $\mathcal{L}_{\mathrm{loc}}$ be the following sheaf on $\mathscr{X}$. For X in $\mathscr{X}$, an element of $\mathcal{L}_{\mathrm{loc}}(X)$ is a triple (p, g, V) where

(i) p is a graphic and étale map from some smooth Y to X ;

(ii) g is a smooth function $Y \to \mathbb{R}$;

(iii) $V \xrightarrow{\omega} Y$ is a $(d+1)$-dimensional Θ-oriented Morse vector bundle.

Conditions: The map $(p, g)\colon Y \to X \times \mathbb{R}$ is proper and $p\omega\colon V \to X$ is a graphic map.

Definition 5.11. An element of $\mathcal{W}^{\mu}_{\mathrm{loc}}(X)$ consists of an element (π, f) of $\mathcal{W}_{\mathrm{loc}}(X)$ with $\pi: E \to X$, an element (p, g, V) of $\mathcal{L}_{\mathrm{loc}}(X)$ with $p: Y \to X$ and a diffeomorphism $Y \to \Sigma(\pi, f)$ over $X \times \mathbb{R}$ covered by a vector bundle isomorphism from V to $T^{\pi}E|\Sigma(\pi, f)$. *Condition*: the vector bundle isomorphism preserves the Θ-orientations and carries the function f_V on V to $w \mapsto \frac{1}{2}H(w, w)$ on $T^{\pi}E|\Sigma(\pi, f)$.

LEMMA 5.12. *The forgetful map* $\mathcal{W}^{\mu}_{\mathrm{loc}} \to \mathcal{W}_{\mathrm{loc}}$ *is a weak equivalence.*

Proof. This is a straightforward application of Proposition 2.18. □

There is also a forgetful map $\mathcal{W}^{\mu}_{\mathrm{loc}} \to \mathcal{L}_{\mathrm{loc}}$. We now describe a homotopy inverse for this. Fix X in $\mathscr{X}$ and let (p, g, V) be an element of $\mathcal{L}_{\mathrm{loc}}(X)$, with $p: Y \to X$. Let $E = V$ and let $\pi: E \to \mathbb{R}$ agree with the composition $V \to Y \to X$. Then $T^{\pi}E$ is identified with $\omega^* V$ (where $\omega: V \to Y$ is the vector bundle projection) and so has a preferred Θ-orientation. Let $f: E \to \mathbb{R}$ be given by

$$f(v) = g(y) + f_V(v), \qquad f_V(v) = \langle v, \varrho v \rangle \tag{5.10}$$

for $y \in Y$ and v in the fiber of V over y. Then (π, f) together with the data (p, g, V) and the identifications $Y \to \Sigma(\pi, f)$ and $V \to T^{\pi}E|\Sigma(\pi, f)$ is an element of $\mathcal{W}^{\mu}_{\mathrm{loc}}(X)$. This defines a map $\mathcal{L}_{\mathrm{loc}}(X) \to \mathcal{W}^{\mu}_{\mathrm{loc}}(X)$.

PROPOSITION 5.13. *The map* $\mathcal{L}_{\mathrm{loc}} \to \mathcal{W}^{\mu}_{\mathrm{loc}}$ *defined above is a weak equivalence*; *consequently the forgetful map* $\mathcal{W}^{\mu}_{\mathrm{loc}} \to \mathcal{L}_{\mathrm{loc}}$ *is also a weak equivalence.*

Proof. We are going to use the relative surjectivity criterion of Proposition 2.18. To deal with the absolute case first, we assume given X in $\mathscr{X}$ and $(\pi, f, p, g, V, ...) \in \mathcal{W}^{\mu}_{\mathrm{loc}}(X)$, with $\pi: E \to \mathbb{R}$ and $f: E \to \mathbb{R}$ and $V \to Y$. Let $\Sigma = \Sigma(\pi, f)$ be the fiberwise singularity set of f. Choose a vertical tubular neighborhood of Σ in E (see Definition 3.7 and Lemma 2.8). As a vector bundle, this is identified with the normal bundle of Σ in E, which is identified with $T^{\pi}E|\Sigma$, hence with $V \to Y$. Therefore we may write $V \subset E$ from now on. By Proposition 3.16, the element $(\pi, f, p, g, V, ...)$ in $\mathcal{W}^{\mu}_{\mathrm{loc}}(X)$ is concordant to $(\pi^{(1)}, f^{(1)}, p, g, V, ...)$ where $\pi^{(1)}$ and $f^{(1)}$ are the restrictions of π and f to $V \subset E$, respectively. The next step is to improve $f^{(1)}$.

Let $\psi: \mathbb{R} \to [0, 1]$ be a smooth nonincreasing function such that $\psi(t) = 1$ for $t < 1 + \varepsilon$ and $\psi(t) = 0$ for $t > 2 - \varepsilon$, for some small $\varepsilon > 0$. For $t \in \mathbb{R}$ let $f^{(t)}$ be given by

$$v \mapsto \begin{cases} fp(v) + \psi(t)^{-2}(f(\psi(t)v) - fp(v)) & \text{for } \psi(t) > 0 \text{ and } v \in V \\ fp(v) + \frac{1}{2}H(pv)(v, v) & \text{for } \psi(t) = 0 \text{ and } v \in V \end{cases}$$

where $H(pv)$ denotes the vertical Hessian of f at $p(v)$. Let $\pi^{(t)} = \pi^{(1)}$ for t in $[1, 2]$. Then $t \mapsto (\pi^{(t)}, f^{(t)})$ defines a concordance from $(\pi^{(1)}, f^{(1)}, p, g, V, ...)$ to

$(\pi^{(2)}, f^{(2)}, p, g, V, \ldots)$. With Θ-orientations aside, $(\pi^{(2)}, f^{(2)}, p, g, V, \ldots)$ clearly lifts to $\mathcal{L}_{\mathrm{loc}}(X)$. In the presence of Θ-orientations, a third concordance is needed to achieve agreement between two Θ-orientations on the vertical tangent bundle of $V \to X$. The two Θ-orientations already agree on the zero section of V (as a vector bundle on Y). Since the inclusion of the zero section of V is a homotopy equivalence, it is easy to find a homotopy between the two Θ-orientations, and this constitutes the third concordance. We have now established the absolute case of the relative surjectivity condition of 2.18 for our map $\mathcal{L}_{\mathrm{loc}} \to \mathcal{W}_{\mathrm{loc}}$. The relative case is not much more difficult and we leave it to the reader. □

We next come to the homotopy colimit decompositions of the right-hand column of (5.1), based on the following key observation.

LEMMA 5.14. *Let $(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$, with $p: Y \to X$. For every $x \in X$ and every $b > 0$ there exists a neighborhood U of x in X such that, on every component of $p^{-1}(U)$, the function g is either bounded below by $-b$ or bounded above by b.*

Proof. Choose a descending sequence of open balls U_i for $i = 0, 1, 2, 3, \ldots$ forming a neighborhood basis for x in X. If the statement is false, then there exists $b > 0$ and connected subsets $K_i \subset Y$ for $i = 0, 1, 2, 3, \ldots$ such that $p(K_i) \subset U_i$ and $g(K_i) \supset [-b, b]$ for all i. Choose $z_i \in K_i$ such that $g(z_i) = 0$. The sequence $z_0, z_1, z_2, \ldots$ in Y must have a convergent (infinite) subsequence, because $(p, g): Y \to X \times \mathbb{R}$ is proper and the two image sequences in X and $\mathbb{R}$ converge. Let $z_\infty \in Y$ be the point which the subsequence converges to. Then $p(z_\infty) = x$ and $g(z_\infty) = 0$. Now $p: Y \to X$ is étale. Hence, for sufficiently large i, there are unique neighborhoods U_i' of z_∞ in Y such that p maps U_i' diffeomorphically to U_i. It follows that $z_i \in U_i'$ for infinitely many i and hence $K_i \subset U_i'$ for infinitely many i. But it is also clear that the diameter of $g(U_i')$ tends to zero as i tends to infinity; hence the lim inf of the diameters of the intervals $g(K_i)$ is zero, which contradicts our assumption. □

Definition 5.15. Fix S in $\mathscr{K}$. We define a sheaf $\mathcal{L}_{\mathrm{loc},S}$ on $\mathscr{X}$. For X in $\mathscr{X}$, an element of $\mathcal{L}_{\mathrm{loc},S}(X)$ is an element (p, g, V) of $\mathcal{L}_{\mathrm{loc}}(X)$, where p has source Y, together with a continuous function $\delta: Y \longrightarrow \{-1, 0, +1\}$, and a diffeomorphism

$$h: S \times X \longrightarrow \delta^{-1}(0) \subset Y$$

over $\{0, 1, \ldots, d+1\} \times X$. *Condition*: Every $x \in X$ has a neighborhood U in X such that g admits a lower bound on $p^{-1}(U) \cap \delta^{-1}(+1)$ and an upper bound on $p^{-1}(U) \cap \delta^{-1}(-1)$.

In Definition 5.15, the function δ clearly has to be constant on each component of Y. Note that the Morse vector bundle structure on $V \to Y$ determines a map $Y \to \{0,1,\dots,d+1\}$ given by the Morse index: $y \mapsto \dim(V_y^{-\varrho})$. This is what we mean when referring to Y as a space over $\{0,1,\dots,d+1\} \times X$.

A morphism $(k,\varepsilon)\colon R \to S$ in $\mathscr{K}$ induces a map $\mathcal{L}_{\mathrm{loc},S} \to \mathcal{L}_{\mathrm{loc},R}$ taking an element (p,g,V,δ,h) of $\mathcal{L}_{\mathrm{loc},S}(X)$ to (p,g,V,δ',h') where $h'(r,x) = h(k(r),x)$ for $(r,x) \in R \times X$ and

$$\delta'(y) = \begin{cases} \varepsilon(s) & \text{if } y = h(s,x) \text{ where } s \in S \smallsetminus k(R),\ x \in X \\ \delta(y) & \text{otherwise.} \end{cases} \tag{5.11}$$

This makes the rule $T \mapsto \mathcal{L}_{\mathrm{loc},T}$ into a contravariant functor from $\mathscr{K}$ to the category of sheaves on $\mathscr{X}$. Moreover, for each T in $\mathscr{K}$ there is a forgetful map $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc}}$, and the maps $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},S}$ induced by morphisms $S \to T$ in $\mathscr{K}$ are over $\mathcal{L}_{\mathrm{loc}}$. This leads to a canonical map of sheaves

$$v\colon \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_{\mathrm{loc},T} \longrightarrow \mathcal{L}_{\mathrm{loc}}\,. \tag{5.12}$$

Proposition 5.16. *The map v in* (5.12) *is a weak equivalence.*

Proof. Let $\mathcal{L}^{\delta}_{\mathrm{loc}}$ be the following sheaf on $\mathscr{X}$ with category structure. An object of $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ is an element (p,g,V) of $\mathcal{L}_{\mathrm{loc}}(X)$, with $p\colon Y \to X$, together with a continuous function $\delta\colon Y \to \{-1,0,+1\}$ subject to the following condition:

> Every $x \in X$ has a neighborhood U in X such that g admits a lower bound on $p^{-1}(U) \cap \delta^{-1}(+1)$, an upper bound on $p^{-1}(U) \cap \delta^{-1}(-1)$, and both an upper and a lower bound on $p^{-1}(U) \cap \delta^{-1}(0)$.

Given two such objects, (p,g,V,δ_a) and (p,g,V,δ_b) with the same underlying (p,g,V), we write $(p,g,V,\delta_a) \le (p,g,V,\delta_b)$ if $\delta_a^{-1}(+1) \subset \delta_b^{-1}(+1)$ and $\delta_a^{-1}(-1) \subset \delta_b^{-1}(-1)$. Then there is a unique morphism from (p,g,V,δ_a) to (p,g,V,δ_b), otherwise there is none. Thus the category $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ is a poset.

The map v in (5.12) can now be factorized as follows:

$$\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_{\mathrm{loc},T} \xrightarrow{v_1} \beta\mathcal{L}^{\delta}_{\mathrm{loc}} \xrightarrow{v_2} \mathcal{L}_{\mathrm{loc}} \tag{5.13}$$

Here v_2 is induced by the forgetful map $\mathcal{L}^{\delta}_{\mathrm{loc}} \to \mathcal{L}_{\mathrm{loc}}$. (Compare Proposition 4.10.) To describe v_1 we recall that $\operatorname{hocolim}_T \mathcal{L}_{\mathrm{loc},T}$ was defined as

$$\beta(\mathscr{K}^{\mathrm{op}}\textstyle\int\mathcal{L}_{\mathrm{loc},\bullet}).$$

For connected X, an object in $(\mathscr{K}^{\mathrm{op}}\int\mathcal{L}_{\mathrm{loc},\bullet})(X)$ consists of an object T in $\mathscr{K}$ and an element a in $\mathcal{L}_{\mathrm{loc},T}(X)$. A morphism from (T,a) to (S,b) is a morphism $S \to T$ in $\mathscr{K}$ taking a to b. An object (T,a) in $(\mathscr{K}^{\mathrm{op}}\int\mathcal{L}_{\mathrm{loc},\bullet})(X)$ with

$a = (p, g, V, \delta, h)$ determines an object (p, g, V, δ) in $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$. This canonical association is a functor, for each X, and as such induces v_1. The next two lemmas complete the proof. □

LEMMA 5.17. *The map v_1 of* (5.13) *is a weak equivalence.*

Proof. For an object (p, g, V, δ) of $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$, the subset $Y_0 = \delta^{-1}(0)$ of Y is closed and $g\colon Y_0 \to \mathbb{R}$ is bounded locally in X. Thus $p\colon Y_0 \to X$ is a proper étale map, hence a covering. The object lifts to $(\mathscr{K}^{\mathrm{op}}\textstyle\int \mathcal{L}_{\mathrm{loc},\bullet})(X)$ if and only if that covering is trivial (a product covering) over each connected component of X. This shows that the functor $(\mathscr{K}^{\mathrm{op}}\textstyle\int \mathcal{L}_{\mathrm{loc},\bullet})(X) \longrightarrow \mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ is always fully faithful, and that it is an equivalence of categories when X is simply connected.

In particular, we have an equivalence of categories for the extended simplices, $X = \Delta^k_e$ where $k \geq 0$. Therefore $|\mathscr{K}^{\mathrm{op}}\textstyle\int \mathcal{L}_{\mathrm{loc},\bullet})| \longrightarrow |\mathcal{L}^{\delta}_{\mathrm{loc}}|$ is a weak homotopy equivalence, cf. Section 4.1. □

LEMMA 5.18. *The map v_2 of* (5.13) *is a weak equivalence.*

Proof. The proof is completely analogous to the proof of Proposition 4.10. We note that given objects (p, g, V, δ_1) and (p, g, V, δ_2) in $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ with the same underlying $(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$, there always exists an object (p, g, V, δ_3) in $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ such that

$$\begin{aligned}(p, g, V, \delta_3) &\leq (p, g, V, \delta_1)\\ (p, g, V, \delta_3) &\leq (p, g, V, \delta_2).\end{aligned}$$

Namely, let $\delta_3(z) = +1$ if and only if $\delta_1(z) = +1 = \delta_2(z)$; let $\delta_3(z) = -1$ if and only if $\delta_1(z) = -1 = \delta_2(z)$, and let $\delta_3(z) = 0$ in the remaining cases.

Now we apply Proposition 2.18 to v_2. Given $(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$, we can by Lemma 5.14 find a locally finite covering of X by open subsets U_j, where $j \in J$, such that $(p, g, V)\,|U_j$ has a lift φ_{jj} to $\mathrm{ob}(\mathcal{L}^{\delta}_{\mathrm{loc}})(U_j)$ for all j. With the observation just above, it is easy to extend the collection of the φ_{jj} to a collection of objects $\varphi_{RR} \in \mathrm{ob}(\mathcal{L}^{\delta}_{\mathrm{loc}})(U_R)$, in such a way that $\varphi_{RR} \leq \varphi_{QQ}|U_R$ whenever $Q \subset R$, giving a morphism in $\mathcal{L}^{\delta}_{\mathrm{loc}}(U_R)$. The collection of these φ_{RR} is then an element of $\beta\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$. This establishes the absolute case of the hypothesis in 2.18, and the verification is much the same in the relative case. □

Definition 5.19. Fix T in $\mathscr{K}$. We define a map from $\mathcal{L}_{\mathrm{loc},T}$ to $\mathcal{W}_{\mathrm{loc},T}$ by

$$\mathcal{L}_{\mathrm{loc},T}(X) \ni (p, g, V, \delta, h) \quad \mapsto \quad h^*(V) \in \mathcal{W}_{\mathrm{loc},T}(X).$$

(Here $h^*(V)$ should be interpreted as the restriction of V to $h(T \times X)$ and the bundle projection should be composed with h^{-1}. Then $h^*(V) \to X$ is graphic.)

There is an equally simple map in the other direction, $\mathcal{W}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},T}$. Indeed, we can identify $\mathcal{W}_{\mathrm{loc},T}(X)$ with the subset of $\mathcal{L}_{\mathrm{loc},T}(X)$ consisting of the elements $(p,g,V,\delta,h) \in \mathcal{L}_{\mathrm{loc},T}(X)$ which have $h = \mathrm{id}_{T\times X}$ and $\delta \equiv 0$, $g \equiv 0$.

Lemma 5.20. *The inclusion $\mathcal{W}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},T}$ is a weak equivalence.*

Proof. We use Proposition 2.18. Given $(p,g,V,\delta,h) \in \mathcal{L}_{\mathrm{loc},T}(X)$ with $p\colon Y \to X$, choose a smooth $\psi\colon [-\infty,1/2[\to [0,\infty[$ such that $\psi(s)=0$ for s close to 0 and $\psi(s)$ tends to $+\infty$ for $s \to 1/2$. Choose another smooth $\varphi\colon \mathbb{R} \to [0,1]$ such that $\varphi(s)=1$ for s close to 0 and $\varphi(s)=0$ for s close to 1. Then define a concordance

$$(\bar p,\bar g,\bar V,\bar\delta,\bar h) \in \mathcal{L}_{\mathrm{loc},T}(X\times\mathbb{R})$$

in the following way. The source of $\bar p$ is the union of $Y\times\,]-\infty,1/2[$ and $h(T\times X)\times\,]0,\infty[$. The formula for $\bar p$ is $\bar p(y,s) = (p(y),s)$. (To ensure that $\bar p$ is graphic, we should define the source of $\bar p$ and $\bar g$ as a subset of the pullback of $p\colon Y \to X$ along the projection $X\times\mathbb{R} \longrightarrow X$. See Definition 2.2.) The formula for $\bar g$ is $\bar g(y,s) := g(y)\cdot\varphi(s)$ if y is in $h(T\times X)$ and $\bar g(y,s) := g(y) + \delta(y)\psi(s)$ otherwise. The vector bundle $\bar V$ is the pullback of V under the projection. The formula for $\bar h$ is $\bar h(t,x,s) := (h(t,x),s)$ and the formula for $\bar\delta$ is $\bar\delta(y,s) = \delta(y)$. By inspection, $(\bar p,\bar g,\bar V,\bar\delta,\bar h)$ is a concordance from $(p,g,V,\delta,h) \in \mathcal{L}_{\mathrm{loc},T}(X)$ to an element $(p',g',V',\delta',h') \in \mathcal{L}_{\mathrm{loc},T}(X)$ where h' is a homeomorphism and $g' \equiv 0$. With some renaming we can arrange h' to be an identity map, so that $(p',g',V',\delta',h') \in \mathcal{W}_{\mathrm{loc},T}(X)$. If a closed subset C of X is given, and the restriction of (p,g,V,δ,h) to some open neighborhood U of C is already in $\mathcal{W}_{\mathrm{loc},T}(U)$, then the concordance just constructed is constant on U, giving the relative surjectivity condition in Proposition 2.18. □

Since the composition $\mathcal{W}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},T} \to \mathcal{W}_{\mathrm{loc},T}$ is the identity, we get

Corollary 5.21. *The map $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{W}_{\mathrm{loc},T}$ of Definition* 5.19 *is a weak equivalence.*

Summarizing, we have established the weak equivalences of the right-hand column of diagram (5.1), and conclude:

Theorem 5.22. *There is a homotopy equivalence*

$$|\mathcal{W}_{\mathrm{loc}}| \simeq \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} |\mathcal{W}_{\mathrm{loc},T}|.$$

5.4. *Upper left-hand column*: *Couplings.*

Definition 5.23. An element of $\mathcal{W}^{\mu}(X)$ is an element $(\pi,f,p,g,V,\dots)$ of $\mathcal{W}^{\mu}_{\mathrm{loc}}(X)$ such that $(\pi,f) \in \mathcal{W}(X)$.

Definition 5.24. A *coupling* between an element (π, f) of $\mathcal{W}(X)$ with $\pi: E \to X$ and an element (p, g, V) of $\mathcal{L}_{\mathrm{loc}}(X)$ with $\omega: V \to Y$ is a smooth embedding $\lambda: \mathrm{sdl}(V, \varrho) \to E$ over X which satisfies $f\lambda(v) = f_V(v) + g(\omega(v))$ for $v \in \mathrm{sdl}(V, \varrho)$, has $\mathrm{im}(\lambda) \supset \Sigma(\pi, f)$ and respects Θ-orientations of the vertical tangent bundles along fiberwise singularity sets.

Remark 5.25. The condition $f\lambda(v) = f_V(v) + g(\omega(v))$ implies that the embedding λ takes the zero section of V to the fiberwise singularity set $\Sigma(\pi, f)$. The condition $\mathrm{im}(\lambda) \supset \Sigma(\pi, f)$ forces an identification of the vector bundle $\omega: V \to Y$ with $T^\pi E|\Sigma(\pi, f) \longrightarrow \Sigma(\pi, f)$. These are the vertical tangent bundles along fiberwise singularity sets referred to in Definition 5.24. Both are Θ-oriented vector bundles.

Remark 5.26. The embedding $\lambda: \mathrm{sdl}(V) \to E$ need not have a closed image, because the étale map $Y \to X$ need not be a closed map. But $\mathrm{im}(\lambda)$ is locally compact, therefore locally closed in E.

Definition 5.27. For X in $\mathscr{X}$, an element of $\mathcal{L}(X)$ is a triple consisting of an element in $\mathcal{W}(X)$, an element in $\mathcal{L}_{\mathrm{loc}}(X)$ and a coupling λ between the two.

PROPOSITION 5.28. *The forgetful map $\mathcal{L} \to \mathcal{W}^{\mu}$ is a weak equivalence.*

Proof. Again we use the relative surjectivity criterion of Proposition 2.18 and again we begin with the absolute case. Fix X in $\mathscr{X}$ and $(\pi, f, p, g, V, ...)$ in $\mathcal{W}^\mu(X)$, with $\pi: E \to X$. We want to lift the concordance class of (π, f) to a class in $\mathcal{L}[X]$. As in the proof of Proposition 5.13, we begin by choosing a vertical tubular neighborhood of $\Sigma = \Sigma(\pi, f)$ in E. As a vector bundle, this is identified with the normal bundle of Σ in E, which is identified with $T^\pi E|\Sigma$, hence with $V \to Y$. From now on we can write $V \subset E$ and $\omega: V \to \Sigma$. By the Morse-Palais lemma [22], we can arrange that $f(v) = f_V(v) + f\omega(v)$ for all v in a neighborhood U of the zero section of V. Without loss of generality, the neighborhood U contains all $v \in \mathrm{sdl}(V, \varrho)$ for which $|f\omega(v)| \leq 1$ and $|f_V(v)| \leq 2$. (If not, replace f by $(\psi\pi) \cdot f$ where $\psi: X \to [1, \infty[$ is a suitable smooth function. Multiply the inner product on V by ψ, too. The elements $(\pi, f, p, g, V, ...)$ and $(\pi, \psi \cdot f, p, (\psi p) \cdot g, V, ...)$ are clearly concordant.)

Now choose a smooth embedding $e: \mathbb{R} \to \mathbb{R}$ with $\mathrm{im}(e) =]-1, 1[$ and $0 < e' \leq 1$ throughout. Then $(\pi, f, p, g, V, ...)$ is concordant to $(\pi^\sharp, f^\sharp, p^\sharp, g^\sharp, V^\sharp, ...)$, where $\pi^\sharp$ is the restriction of π to $E^\sharp = f^{-1}(\mathrm{im}(e))$ and $f^\sharp$ is $e^{-1}f$ on $E^\sharp$. Let $\Sigma^\sharp = \Sigma \cap E^\sharp$ and $V^\sharp = V|\Sigma^\sharp$. Let

$$K = \left\{ v \in \mathrm{sdl}(V^\sharp, \varrho) \;\middle|\; |f_V(v) + f\omega(v)| < 1 \right\}.$$

For $v \in K$ we have $|f\omega(v)| < 1$ and $|f_V(v)| < 2$, so $K \subset U$ by our assumptions and consequently $f|K = f_V|K + f\omega|K$. It follows that $K \subset E^\sharp$. Using

Proposition 5.9, but writing λ for τ, we can construct an embedding

$$\lambda\colon \mathrm{sdl}(V^\sharp, \varrho) \longrightarrow K$$

relative to and over $\Sigma^\sharp$, such that

$$(f_V + f\omega) \circ \lambda = e \circ (f_V + e^{-1} f\omega) = e \circ (f_V + f^\sharp\omega)\,.$$

This can also be viewed as an embedding of $\mathrm{sdl}(V^\sharp, \varrho)$ in $E^\sharp$. We have

$$f^\sharp\lambda = e^{-1} f\lambda = e^{-1}(f_V + f\omega)\lambda = e^{-1}e(f_V + f^\sharp\omega) = f_V + f^\sharp\omega$$

on $\mathrm{sdl}(V^\sharp, \varrho)$. That is, λ is a coupling, in the sense of Definition 5.24, of $(\pi^\sharp, f^\sharp) \in \mathcal{W}(X)$ with $(p, g, V^\sharp) \in \mathcal{L}_{\mathrm{loc}}(X)$ where $p = \pi^\sharp|\Sigma^\sharp$ and $g = f^\sharp|\Sigma^\sharp$. Note that λ identifies $V^\sharp$ with $T^\pi E|\Sigma^\sharp$, as explained in the remarks following Definition 5.24, so that $V^\sharp$ inherits a Morse vector bundle structure and a Θ-orientation from $T^\pi E|\Sigma^\sharp$. (The new Morse structure on $V^\sharp$ does not quite agree with the restriction of the Morse structure on V which we used earlier in this proof. In fact the two riemannian structures agree up to a scalar factor given by a strictly positive function $\Sigma^\sharp \to \mathbb{R}$.) The coupling λ promotes the element $(\pi^\sharp, f^\sharp, p^\sharp, g^\sharp, V^\sharp, ...)$ to an element of $\mathcal{L}(X)$, except for the matter of Θ-orientations which we can handle as in the proof of Proposition 5.13. This establishes the absolute case of the relative surjectivity condition.

The relative case is only slightly more difficult. We sketch it. Again fix X in $\mathscr{X}$ and $(\pi, f, p, g, V, ...)$ in $\mathcal{W}^\mu(X)$, with $\pi\colon E \to X$. Let $C \subset X$ be closed. We want to find an element in $\mathcal{L}(X)$ whose image in $\mathcal{W}(X)$ is concordant rel C to $(\pi, f, p, g, V, ...)$. This can be constructed as in the absolute case, except for one small change which consists in replacing the embedding $e\colon \mathbb{R} \to \mathbb{R}$ above by a smooth family of smooth embeddings $e_x\colon \mathbb{R} \to \mathbb{R}$, depending on $x \in X$. Then we have the option to choose $e_x = \mathrm{id}_{\mathbb{R}}$ for x in a small neighborhood of C, while having $\mathrm{im}(e_x) =]-1, 1[$ for x outside a slightly larger neighborhood of C. ∎

The forgetful map $\mathcal{L} \to \mathcal{L}_{\mathrm{loc}}$ is not surjective in general, nor does it have the concordance lifting property. However certain "easy" concordances in $\mathcal{L}_{\mathrm{loc}}$ can be lifted across the forgetful map $\mathcal{L} \to \mathcal{L}_{\mathrm{loc}}$, and this fact will be needed later.

Lemma 5.29. *Let $(\pi, f, p, g, V, \lambda)$ be an element of $\mathcal{L}(X)$. Let $(\bar p, \bar g, \bar V)$ in $\mathcal{L}_{\mathrm{loc}}(X \times \mathbb{R})$ be a concordance whose initial position is $(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$. Suppose there exists a pullback diagram*

$$\begin{tikzcd} \bar Y \arrow[r] \arrow[d, "\bar p"] & Y \arrow[d, "p"] \\ X \times \mathbb{R} \arrow[r, "\text{proj.}"] & X \end{tikzcd}$$

where the map $\bar{Y} \to Y$ *restricts to* $\mathrm{id}\colon Y \to Y$ *over* $X \cong X \times 0$. *Then* $(\bar{p}, \bar{g}, \bar{V})$ *lifts to a concordance* $(\bar{\pi}, \bar{f}, \dots) \in \mathcal{L}(X \times \mathbb{R})$ *whose initial position is* $(\pi, f, p, g, V, \lambda) \in \mathcal{L}_{\mathrm{loc}}(X)$. *If the concordance* $(\bar{p}, \bar{g}, \bar{V})$ *is relative to a closed subset* A *of* X, *then* $(\bar{\pi}, \bar{f}, \dots)$ *can also be taken relative to* A.

Proof. The statement involves $\mathbb{R}$ in two ways: as a target for functions such as f and g, and as a time-like axis which parametrizes concordances. To reduce confusion, we will write $\mathbb{R}_\tau$ instead of $\mathbb{R}$ if we mean the time axis.

The restriction of (π, f) to $\partial(\mathrm{im}(\lambda))$ is a submersion $\partial(\mathrm{im}(\lambda)) \to X \times \mathbb{R}$. This follows from the equation $f\lambda = f_V + g\omega$ (where $\omega\colon V \to Y$ is the projection) and either (5.3) or (5.4).

It is therefore possible to find an outward collar for $\mathrm{im}(\lambda)$ in E (the source of π) which is "over" $X \times \mathbb{R}$. We mean by that a smooth codimension zero embedding u of $\partial(\mathrm{im}(\lambda)) \times [0,1]$ in $E \smallsetminus \mathrm{int}(\mathrm{im}(\lambda))$ which extends the inclusion of $\partial(\mathrm{im}(\lambda)) \cong \partial(\mathrm{im}(\lambda)) \times \{1\}$, and which satisfies $\pi(u(z,t)) = \pi(u(z,1))$ as well as $f(u(z,t)) = f(u(z,1))$ for all $z \in \partial(\mathrm{im}(\lambda))$ and $t \in [0,1]$. Note that $u(\partial(\mathrm{im}(\lambda)) \times \{0\})$ is the far end of the collar.

We now construct our concordance $(\bar{\pi}, \bar{f}, \dots)$ as follows. Let $\bar{E} = E \times \mathbb{R}_\tau$ and let $\bar{\pi} = \pi \times \mathrm{id}\colon \bar{E} \to X \times \mathbb{R}_\tau$. Elements of $\bar{E}$ should be relabelled to ensure that $\bar{\pi}$ is graphic, but we will not pay much attention to that now. Since $\bar{Y}$ is identified with $Y \times \mathbb{R}_\tau$, we may also identify $\bar{V}$ with $V \times \mathbb{R}_\tau$, so that $\bar{\omega}\colon \bar{V} \to \bar{Y}$ is identified with $\omega \times \mathrm{id}\colon V \times \mathbb{R}_\tau \to Y \times \mathbb{R}_\tau$. Now we can define $\bar{\lambda}$ by $\bar{\lambda}(v,t) = (\lambda(v), t) \in E \times \mathbb{R}_\tau = \bar{E}$. We then choose a Θ-orientation on the vertical tangent bundle of $\bar{E}$ which agrees with the prescribed Θ-orientations over $E \times 0 \subset \bar{E}$ and the image of $\bar{\lambda}$. Since the inclusion of $(E \times 0) \cup \mathrm{im}(\bar{\lambda})$ in $\bar{E}$ is a homotopy equivalence, this can be done.

It remains to define $\bar{f}$ on $\bar{E}$. For $z \in E$ outside $\mathrm{im}(\lambda) \cup \mathrm{im}(u)$ and any $t \in \mathbb{R}_\tau$ we let $\bar{f}(z,t) = f(z)$. For $z = \lambda(v) \in \mathrm{im}(\lambda)$ we must define

$$\bar{f}(z,t) = f_V(v) + \bar{g}(\omega(v), t) = f(z) + \bar{g}(\omega(v), t) - g(\omega(v)).$$

This leaves the case $z \in \mathrm{im}(u)$, say $z = u(\lambda(v), s)$ with $v \in \partial(\mathrm{sdl}(V, \varrho))$ and $s \in [0,1]$. In that case we say $\bar{f}(z,t) = f(z) + \bar{g}(\omega(v), \psi(s)t) - g(\omega(v))$, using a smooth function $\psi\colon [0,1] \to [0,1]$ which has $\psi(s) = 0$ for s near 0 and $\psi(s) = 1$ for s near 1. $\square$

Definition 5.30. For T in $\mathscr{K}$, we define a sheaf $\mathcal{L}'_T$ as the pullback of

$$\mathcal{L} \xrightarrow{\text{forget}} \mathcal{L}_{\mathrm{loc}} \xleftarrow{\text{forget}} \mathcal{L}_{\mathrm{loc},T}\,.$$

The forgetful maps $\mathcal{L}'_T \to \mathcal{L}$ for T in $\mathscr{K}$ determine a canonical map u from the sheaf $\mathrm{hocolim}_T\, \mathcal{L}'_T$ to $\mathcal{L}$.

Proposition 5.31. *The map* $u\colon \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}'_T \longrightarrow \mathcal{L}$ *is a weak equivalence.*

Proof. The proof is completely analogous to that of Proposition 5.16. There is a factorization of u having the form

$$\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}'_T \xrightarrow{u_1} \beta\mathcal{L}^\delta \xrightarrow{u_2} \mathcal{L} \tag{5.14}$$

where $\mathcal{L}^\delta$ is defined as the pullback of $\mathcal{L} \longrightarrow \mathcal{L}_{\text{loc}} \longleftarrow \mathcal{L}^\delta_{\text{loc}}$. One shows that u_1 and u_2 are weak equivalences. □

5.5. *Lower left-hand column*: *Regularization.* In order to make this section more accessible, we assume to begin with that $\Theta = \star$ and discuss the general case afterwards.

Let $(\pi, f, p, g, V, \delta, h, \lambda)$ be an element of $\mathcal{L}'_T(X)$ with

$$\begin{aligned} &(\pi, f)\colon E \to X \times \mathbb{R}\,, \quad (p, g)\colon Y \to X \times \mathbb{R}\,, \quad V \xrightarrow{\omega} Y\,, \\ &\delta\colon Y \to \{-1, 0, +1\}\,, \quad h\colon T \times X \cong \delta^{-1}(0)\,, \quad \lambda\colon \mathrm{sdl}(V, \varrho) \longrightarrow E\,. \end{aligned} \tag{5.15}$$

We adopt the notation $Y_+ = \delta^{-1}(+1)$, $Y_- = \delta^{-1}(-1)$, $Y_0 = \delta^{-1}(0)$ and let V_+, V_-, V_0 be the restrictions of the Morse bundle V to these three (open and closed) subspaces of Y.

Definition 5.32. $\mathcal{L}_T$ is the subsheaf of $\mathcal{L}'_T$ consisting of the elements $(\pi, f, p, g, V, \delta, h, \lambda)$ as above with $g|Y_0 \equiv 0$.

Proposition 5.33. *The inclusion* $\mathcal{L}_T \to \mathcal{L}'_T$ *is a weak equivalence.*

Proof. This is a direct application of Proposition 2.18 in conjunction with Lemma 5.29. □

For an element $(\pi, f, \dots)$ of $\mathcal{L}_T(X)$ as above we define the regularization $(\pi^{\text{rg}}, f^{\text{rg}})$ with $\pi^{\text{rg}}\colon E^{\text{rg}} \to X$ and $f^{\text{rg}}\colon E^{\text{rg}} \to \mathbb{R}$ by

$$\begin{aligned} E^{\text{rg}} &= E \smallsetminus \lambda(V_+^\varrho \cup V_0^\varrho \cup V_-^{-\varrho})\,, \\ \pi^{\text{rg}} &= \pi | E^{\text{rg}}\,, \\ f^{\text{rg}}(z) &= \begin{cases} f(z) & \text{if } v \notin \mathrm{im}(\lambda) \\ f_V^+(v) & \text{if } z = \lambda(v) \text{ and } v \in V_+ \cup V_0 \\ f_V^-(v) & \text{if } z = \lambda(v) \text{ and } v \in V_-\ . \end{cases} \end{aligned} \tag{5.16}$$

The maps f_V^+ and f_V^- were defined in (5.8). We note that E^{rg} is an open subset of E despite Remark 5.26. (The condition on lower and upper bounds in Definition 5.15 ensures that $\lambda(V_+^\varrho)$, $\lambda(V_-^{-\varrho})$, $\lambda(V_0^\varrho)$ and $\lambda(V_0^{-\varrho})$ are closed subsets of E.) Moreover, $\pi^{\mathrm{rg}}\colon E^{\mathrm{rg}} \to X$ is a smooth submersion and f^{rg} is regular on each fiber of π^{rg}. Hence

$$(\pi^{\mathrm{rg}}, f^{\mathrm{rg}})\colon E^{\mathrm{rg}} \longrightarrow X \times \mathbb{R}$$

is a smooth proper submersion, and by Ehresmann's fibration lemma we have

PROPOSITION 5.34. *The map $(\pi^{\mathrm{rg}}, f^{\mathrm{rg}})\colon E^{\mathrm{rg}} \longrightarrow X \times \mathbb{R}$ is a smooth bundle of closed d-manifolds.*

It follows that the inverse image of 0 under f^{rg} is a bundle $q\colon M \to X$ of closed d-manifolds. Since the restriction of $f^{\mathrm{rg}} \circ \lambda$ to $\mathrm{sdl}(V_0, \varrho)$ is $f^+_{V_0}$, the restriction of λ gives an embedding

$$e\colon (f^+_{V_0})^{-1}(0) \quad \longrightarrow \quad M\ . \tag{5.17}$$

The source of e is identified with $D(V_0^{\varrho}) \times_{T\times X} S(V_0^{-\varrho})$ by formula (5.9). The diffeomorphism $h\colon T \times X \to Y_0$ gives the required element $h^*(V_0) \in \mathcal{W}_{\mathrm{loc},T}(X)$.

Starting from an element in $\mathcal{L}_T(X)$, we have now produced an element of $\mathcal{W}_T(X)$ consisting of $q\colon M \to X$ and the embedding e.

It is convenient to introduce two subsheaves $\mathcal{L}^!_T$ and $\mathcal{L}^{!!}_T$ of $\mathcal{L}_T$. For $\mathcal{L}^!_T$ we add to the data in 5.32 the condition that $g \geq 1$ on Y_+ and $g \leq -1$ on Y_-. For the sheaf $\mathcal{L}^{!!}_T$ we add the stronger condition that $\delta \equiv 0$, so that Y_+ and Y_- are empty (and $g \equiv 0$).

LEMMA 5.35. *The inclusions $\mathcal{L}^!_T \to \mathcal{L}_T$ and $\mathcal{L}^{!!}_T \to \mathcal{L}^!_T$ are weak equivalences.*

Proof. A direct application of 5.29 shows that the inclusion $\mathcal{L}^!_T \to \mathcal{L}_T$ is a weak equivalence. For the inclusion $\mathcal{L}^{!!}_T \to \mathcal{L}^!_T$ we use Lemma 2.19. Given an element $(\pi, f, \dots)$ of $\mathcal{L}^!_T(X)$ as in (5.15), we choose a suitable smooth $e\colon X \times \mathbb{R} \to \mathbb{R}$ such that each $e_x\colon \mathbb{R} \to \mathbb{R}$ defined by $e_x(t) = e(x,t)$ is a smooth orientation-preserving embedding, with $e_x(0) = 0$. In addition we require that $0 < e'_x \leq 1$ for all $x \in X$, with a view to Proposition 5.9, and that the image of e_x does not contain any nonzero critical values of the Morse function $f|E_x$. (For example, if $-1 < e_x < 1$, then $\mathrm{im}(e_x)$ does not contain any nonzero critical values of $f|E_x$.) Define $E^{(1)} \subset E$, $\pi^{(1)}$ and $f^{(1)}$ exactly as in Lemma 2.19. Let $V^{(1)} = V_0$. Define $\lambda^{(1)}\colon \mathrm{sdl}(V^{(1)}, \varrho) \to E^{(1)}$ by composing $\lambda\colon \mathrm{sdl}(V, \varrho) \to E$ with an embedding $\tau\colon \mathrm{sdl}(V_0, \varrho) \to \mathrm{sdl}(V_0, \varrho)$ over X as in Proposition 5.9, so that $f_V(\tau(v)) = e_x(f_V(v))$ for $x \in X$, $y \in Y_0$ with $p(y) = x$ and $v \in V_y$. We get an element $(\pi^{(1)}, f^{(1)}, \dots)$ of $\mathcal{L}^{!!}_T(X)$ which is concordant in $\mathcal{L}^!_T$ to $(\pi, f, \dots)$. Therefore $\mathcal{L}^{!!}_T[X] \to \mathcal{L}^!_T[X]$ is surjective. The same argument gives surjectivity in the relative case, $\mathcal{L}^{!!}_T[X, A; s] \to \mathcal{L}^!_T[X, A; s]$, assuming $A \subset X$ is closed and $s \in \mathrm{colim}_U \mathcal{L}^!_T(U)$ where U runs over the open neighborhoods of A in X. The only detail to watch here is that we need $e_x = \mathrm{id}_{\mathbb{R}}$ for x in a sufficiently small neighborhood of A. We complete the proof by applying Proposition 2.18. $\square$

Proposition 5.36. *The map $\mathcal{L}_T \to \mathcal{W}_T$ defined above is a weak equivalence.*

Proof. By Lemma 5.35, it is enough to verify that the composition

$$\mathcal{L}_T^{!!} \to \mathcal{L}_T \to \mathcal{W}_T$$

is a weak equivalence. But this is almost obvious from Section 5.2. Namely, the long trace construction gives us a map of concordance sets $\mathcal{W}_T[X] \to \mathcal{L}_T^{!!}[X]$ which is inverse to $\mathcal{L}_T^{!!}[X] \to \mathcal{W}_T[X]$. This works equally well in a relative setting, so that Proposition 2.18 can be used. The only thing to watch here is the Θ-orientation issue. For this, fix an element (q, V, e) of $\mathcal{W}_T(X)$ with $q: M \to X$ and write

$$e: (f_V^+)^{-1}(0) \quad \longrightarrow \quad M\,.$$

Let E be the long trace of e, with projection $\pi: E \to X$. Then E contains a copy of $C = M \sqcup_{\mathrm{im}(e)} \mathrm{sdl}(V, \varrho)$, where $\mathrm{im}(e)$ is identified with $(f_V^+)^{-1}(0)$. A Θ-orientation of $T^\pi E|C$ is already specified. The inclusion $C \to E$ is a homotopy equivalence, so that there is no obstruction to extending the Θ-orientation of $T^\pi E|C$ to a Θ-orientation of $T^\pi E$. □

Remark 5.37. Some of the constructions above involve choices of pushouts in the category of sets. These choices can be fixed in advance to ensure that $T \mapsto \mathcal{W}_T$ really is a functor. For example, adding the following to Definition 5.3 turns out to be enough: a choice of push-out $M \sqcup_{\mathrm{im}(e)} \mathrm{sdl}(V, \varrho)$ (in the category of sets) with graphic projection to X.

We end the section with the promised discussion of Θ-orientations. We start again with the data list (5.15) for an element of $\mathcal{L}'_T(X)$. The coupling λ identifies $T^\pi E|\mathrm{im}(\lambda)$ with $\omega^* V|\mathrm{sdl}(V, \varrho)$. The differential

$$d^\pi f: T^\pi E \longrightarrow f^*(T\mathbb{R})$$

is surjective over $E \smallsetminus \Sigma \subset E$, where $\Sigma = \Sigma(\pi, f)$. Over $\mathrm{im}(\lambda) \smallsetminus \Sigma$ it has a preferred splitting, since $T^\pi E|\mathrm{im}(\lambda)$ is a Riemannian vector bundle. We redefine $\mathcal{L}_T$ by adding the following two items to Definition 5.32:

(A) A vector bundle splitting of $df: T^\pi E|E \smallsetminus \Sigma \to f^*(T\mathbb{R})|E \smallsetminus \Sigma$ which extends the preferred splitting over $\mathrm{im}(\lambda) \smallsetminus \Sigma$;

(B) The condition that λ (and its fiberwise differential) preserve the given Θ-orientations of the vertical tangent bundles, and not just their restrictions to the fiberwise singularity sets as in Definition 5.24.

Proposition 5.38. *The forgetful map $\mathcal{L}_T \to \mathcal{L}'_T$ is a weak equivalence.*

Proof. Write the map as a composition of two maps, say $\mathcal{L}_T \to \mathcal{L}_T^\natural \to \mathcal{L}'_T$ where $\mathcal{L}_T^\natural$ is the old $\mathcal{L}_T$ as in Definition 5.32, a subsheaf of $\mathcal{L}'_T$. Proposition 2.18 makes it straightforward to verify that

$$\mathcal{L}_T \to \mathcal{L}_T^\natural$$

is a weak equivalence. Proposition 2.18 and Lemma 5.29 together imply that $\mathcal{L}_T^\natural \to \mathcal{L}'_T$ is a weak equivalence. □

Given an element of $\mathcal{L}_T(X)$, consisting of data as in (5.15) and items (A) and (B) just above, we produce a d-manifold bundle $q\colon M \to X$ and an embedding

$$e\colon D(V_0^\varrho) \times_{T\times X} S(V_0^{-\varrho}) \longrightarrow M$$

as before. We note that $T^\pi E|E^{\mathrm{rg}}$ has a canonical splitting,

$$T^\pi E|E^{\mathrm{rg}} \quad\cong\quad \ker(df^{\mathrm{rg}}) \times \mathbb{R}\,.$$

Indeed, over points $z \in E^{\mathrm{rg}}$ not in $\mathrm{im}(\lambda)$ we can use the data of item (A) and over points $z \in \mathrm{im}(\lambda) \cap E^{\mathrm{rg}}$ we can use the Riemannian vector bundle structure on the fiberwise tangent bundle of $\mathrm{im}(\lambda) \to X$. The matching condition in (A) ensures that this gives a continuous splitting. Since $M \subset E^{\mathrm{rg}}$, we deduce a canonical vector bundle splitting

$$T^\pi E|M \quad\cong\quad T^q M \times \mathbb{R}\,.$$

The Θ-orientation on $T^\pi E$ therefore induces a Θ-orientation on $T^qM \times \mathbb{R}$, which amounts to a Θ-orientation on T^qM itself.

In the same way, the codimension 1 inclusion of

$$\{v \in \mathrm{sdl}(V_0, \varrho) \mid f_V^+(v) = 0\}$$

in $\mathrm{sdl}(V, \varrho)$ with preferred normal line bundle leads to a Θ-orientation on the vertical tangent bundle of

$$\{v \in \mathrm{sdl}(V_0, \varrho) \mid f_V^+(v) = 0\} \;\cong\; D(V_0^\varrho) \times_{T\times X} S(V_0^{-\varrho})\,.$$

This is our standard choice of a Θ-orientation on the vertical tangent bundle of the source of e. With this choice e clearly respects the Θ-orientations. Hence (q, V_0, e) is a triple satisfying the requirements for an element of $\mathcal{W}_T(X)$ in Definition 5.3. The instructions of Remark 5.37 for making $T \mapsto \mathcal{W}_T$ into a functor carry over without change to the case of general Θ-orientations.

PROPOSITION 5.39. *The map $\mathcal{L}_T \to \mathcal{W}_T$ defined above is a weak equivalence.*

Proof. The proof of Proposition 5.36 goes through. □

This completes the construction of diagram (5.1) in the general case.

5.6. *The concordance lifting property.*

LEMMA 5.40. *For fixed T in $\mathscr{K}$, the forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ has the concordance lifting property.*

Proof. We first consider the easier case where $\Theta = \star$. Suppose the concordance of $z \in \mathcal{W}_{\mathrm{loc},T}(X)$ is given by the Morse vector bundle $V \to T \times X \times \mathbb{R}$, and let $W = V \mid T \times X \times 0$. There is an isomorphism $j \colon V \to W \times \mathbb{R}$ that restricts to the identity over $X \times T \times]-\infty, \varepsilon[$ and is constant over $X \times T \times]1-\varepsilon, \infty[$. If $(M, q, e) \in \mathcal{W}_T(X)$ lifts z, then $(M \times \mathbb{R}, q \times \mathbb{R}, \hat{e})$ with

$$\hat{e} \colon D(V^{\varrho}) \times_{T \times X \times \mathbb{R}} S(V^{-\varrho}) \xrightarrow{\ j\ } D(W^{\varrho}) \times_{T \times X} S(W^{-\varrho}) \times \mathbb{R} \xrightarrow{e \times \mathbb{R}} M \times \mathbb{R}$$

is a lifting of the concordance.

In the general case, with Θ-orientations, we begin with the construction of a lifted concordance as above, first without worrying about tangential Θ-orientations. We then have to make a choice of Θ-orientation on the fiberwise tangent bundle of a manifold bundle of the form

$$q \times \mathbb{R} \colon M \times \mathbb{R} \longrightarrow X \times \mathbb{R}\,.$$

This is prescribed over the union of U and $\mathrm{im}(\hat{e})$, where U is a neighborhood (germ) of $M \times]-\infty, 0]$ and e is an embedding as in Definition 5.3. Since the inclusion of

$$M \times]-\infty, 0] \ \cup\ \mathrm{im}(\hat{e})$$

in $M \times \mathbb{R}$ is a homotopy equivalence, the problem can be solved. □

Now fix an element (V, ϱ) in $\mathcal{W}_{\mathrm{loc},T}(\star)$. That is, V is a $(d+1)$-dimensional Θ-oriented Morse vector bundle on T with $\dim(V_t{}^{-\varrho})$ equal to the label of t in $\{0, 1, \ldots, d+1\}$. The following is true by definition and trivial reformulations.

LEMMA 5.41. *The fiber of the forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ over (V, ϱ) in $\mathcal{W}_{\mathrm{loc},T}(\star)$ is weakly equivalent to the sheaf which takes an X in $\mathscr{X}$ to the set of all pairs (q, e) where*

(i) *$q \colon M \to X$ is a smooth graphic bundle of closed d-manifolds with a Θ-orientation of the vertical tangent bundle T^qM;*

(ii) *$e \colon D(V^{\varrho}) \times_T S(V^{-\varrho}) \times X \longrightarrow M$ is a smooth embedding over X which is fiberwise Θ-orientation preserving.* □

COROLLARY 5.42. *The fiber of the forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ over $(V, \varrho) \in \mathcal{W}_{\mathrm{loc},T}(\star)$ is weakly equivalent to the sheaf which takes an X in $\mathscr{X}$ to* *the set of all smooth graphic bundles* $q: M \to X$ *of tangentially* Θ*-oriented compact* d*-manifolds with collared boundary, where the boundary bundle* $\partial M \to X$ *is identified with*

$$-(S(V^{\varrho}) \times_T S(V^{-\varrho})) \times X \quad \longrightarrow \quad X.$$

(*The minus sign indicates the* "*opposite*" Θ*-orientation*; *see Remark* 5.43.)

Proof. One removes the interior of $\mathrm{im}(e)$ to get from the triple (M, q, e) in Lemma 5.41 to the kind of bundle described in the corollary. This process is clearly reversible. ☐

Remark 5.43. For a vector bundle $W \to B$ of dimension k with a Θ-orientation, i.e., a section σ of $(\mathrm{Fr}(W) \times \Theta)/\mathrm{GL}(k) \to B$, the opposite Θ-orientation $-\sigma$ can be defined as follows. Compose σ with the action of $r: \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ on Θ, where $r(x_1, x_2, x_3, \ldots, x_k) = (x_1, x_2, x_3, \ldots, -x_k)$.

Remark 5.44. The description of the (homotopy) fiber in Corollary 5.42 uses only the part of T lying over $\{1, 2, \ldots, d\} \subset \{0, 1, 2, \ldots, d, d+1\}$, since spheres of dimension -1 are empty.

5.7. *Introducing boundaries.* Here we are concerned with a slight generalization of diagram (5.1). It is obtained by replacing all families of $(d+1)$-manifolds in sight by families of $(d+1)$-manifolds with a prescribed boundary. For the purposes of this section we indicate the change by a superscript "∂" as in ${}^{\partial}\mathcal{W}$; later, in Sections 6 and 7, the superscript will be dropped.

We assume $d > 0$ and fix a closed nonempty smooth $(d-1)$-manifold C with a Θ-orientation of the tangent bundle TC. The "prescribed boundary" which we have in mind will be $C \times \mathbb{R}$. We assume also that C is nullbordant in the following sense: there exists a compact smooth d-manifold K with collared boundary $\partial K = C$ and a Θ-orientation of TK which extends the specified one on $TC \times \mathbb{R} \cong TK|C$. (Here we use the outward normal field along C to identify $TC \times \mathbb{R}$ with $TK|C$.) For example, C could be S^{d-1} and K could be D^d, with suitable Θ-orientations.

Definition 5.45. An element of ${}^{\partial}\mathcal{W}(X)$ is a pair (π, f) as in 2.7, with $\pi: E \to X$ and $f: E \to \mathbb{R}$, except for the following: we require that a diffeomorphism germ over $X \times \mathbb{R}$ be specified which identifies a neighborhood of ∂E in E with a neighborhood of $X \times C \times 0 \times \mathbb{R}$ in $X \times C \times [0, \infty[\times \mathbb{R}$, respecting the Θ-orientations.

The same change made in the definitions of the sheaves $\mathcal{W}_{\mathrm{loc}}$, $\mathcal{W}^{\mu}$, $\mathcal{W}^{\mu}_{\mathrm{loc}}$, $\mathcal{L}$ and $\mathcal{L}_T$ produces ${}^{\partial}\mathcal{W}_{\mathrm{loc}}$, ${}^{\partial}\mathcal{W}^{\mu}$, ${}^{\partial}\mathcal{W}^{\mu}_{\mathrm{loc}}$ ${}^{\partial}\mathcal{L}$ and ${}^{\partial}\mathcal{L}_T$, respectively. (There is also a small change in Definition 5.24: we require $\mathrm{im}(\lambda) \cap \partial E = \emptyset$.) No changes are needed in the definitions of $\mathcal{L}_{\mathrm{loc}}$ and $\mathcal{L}_{\mathrm{loc},T}$; that is, we put ${}^{\partial}\mathcal{L}_{\mathrm{loc}} = \mathcal{L}_{\mathrm{loc}}$ and ${}^{\partial}\mathcal{L}_{\mathrm{loc},T} = \mathcal{L}_{\mathrm{loc},T}$. The case of ${}^{\partial}\mathcal{W}_T$ is slightly different:

Definition 5.46. An element of ${}^{\partial}\mathcal{W}_T(X)$ is a triple (q, V, e) as in Definition 5.3, except for the following. We require a diffeomorphism germ over X which identifies a neighborhood of ∂M in M with a neighborhood of $X \times C \times \{0\}$ in $X \times C \times [0, \infty[$, respecting the Θ-orientations. We require $\mathrm{im}(e) \cap \partial M = \emptyset$.

The ∂-variant of diagram (5.1) is

(5.18)
$$
\begin{tikzcd}
{}^{\partial}\mathcal{W} \arrow[r] & {}^{\partial}\mathcal{W}_{\mathrm{loc}} \\
{}^{\partial}\mathcal{W}^{\mu} \arrow[r] \arrow[u, "\simeq"'] & {}^{\partial}\mathcal{W}^{\mu}_{\mathrm{loc}} \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] \\
{}^{\partial}\mathcal{L} \arrow[r] \arrow[u, "\simeq"'] & \mathcal{L}_{\mathrm{loc}} \\
\underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ {}^{\partial}\mathcal{L}_T \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] & \underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ \mathcal{L}_{\mathrm{loc},T} \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] \\
\underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ {}^{\partial}\mathcal{W}_T \arrow[r] & \underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ \mathcal{W}_{\mathrm{loc},T}\,.
\end{tikzcd}
$$

To prove that all the maps labelled "$\simeq$" are indeed weak equivalences, one could proceed roughly as in the no-boundary situation. Another method is to show that diagrams (5.18) and (5.1) can be related by a chain of natural transformations all of which are weak equivalences. We now explain how this works for the top left-hand terms, and give some indications for the remaining terms.

The first thing we need to know is that ${}^{\partial}\mathcal{W}(\star)$ is nonempty. Indeed, an element in ${}^{\partial}\mathcal{W}(\star)$ is given by $K \times \mathbb{R}$, where K is the nullbordism for C mentioned earlier, with the projection map $K \times \mathbb{R} \to \mathbb{R}$ and a Θ-orientation on $TK \times T\mathbb{R}$ which can be described as the opposite of the one specified earlier. (It has to extend the preferred Θ-orientation on $TC \times \mathbb{R} \times T\mathbb{R}$ under the identification $TC \times \mathbb{R} \cong TK|C$ which is determined by the *inward* normal field along C. See also Remark 5.43.)

As in the proof of Theorem 3.8, the formula

$$((\pi, f), (\psi, g)) \mapsto (\pi \sqcup \psi, f \sqcup g)$$

defines maps $\mathcal{W} \times \mathcal{W} \to \mathcal{W}$ and $\mathcal{W} \times {}^{\partial}\mathcal{W} \to {}^{\partial}\mathcal{W}$. From Section 4 we know that $|\mathcal{W}| \simeq |h\mathcal{W}|$ and from Section 3 we know that the monoid $\pi_0|h\mathcal{W}|$ is a group. It follows that $\pi_0\mathcal{W} = \mathcal{W}[\star]$ is a group under the above addition law.

We now claim that for fixed $z \in {}^{\partial}\mathcal{W}(*)$, the restriction of the action map to $\mathcal{W} \times z$ is a weak equivalence

$$u\colon \mathcal{W} \times z \to {}^{\partial}\mathcal{W}\,.$$

Indeed the inverse map, essentially from ${}^{\partial}\mathcal{W}$ to $\mathcal{W}$, is given by gluing in the "nullbordism" $K \times \mathbb{R}$. More precisely, we define

$$v\colon {}^{\partial}\mathcal{W} \to \mathcal{W}$$

by taking $(\pi, f) \in {}^{\partial}\mathcal{W}(X)$ with $\pi\colon E \to \mathbb{R}$ to $(\pi \sqcup p_X, f \sqcup p_{\mathbb{R}}) \in \mathcal{W}(X)$, where p_X and $p_{\mathbb{R}}$ are the projections from $X \times K \times \mathbb{R}$ to X and $\mathbb{R}$, respectively. The common source of $\pi \sqcup p_X$ and $f \sqcup p_{\mathbb{R}}$ is the pushout $E \sqcup_{X \times C \times \mathbb{R}} (X \times K \times \mathbb{R})$.

Taking representing spaces, we obtain $|u|$ from $|\mathcal{W} \times z| \simeq |\mathcal{W}|$ to $|{}^{\partial}\mathcal{W}|$ and $|v|$ from $|{}^{\partial}\mathcal{W}|$ to $|\mathcal{W}|$, both well defined up to homotopy. It is easy to verify that the homotopy classes of $|v||u|$ and $|u||v|$ are both given by translation with the concordance class $[v(z)] \in \mathcal{W}[\star]$. Since $\mathcal{W}[\star]$ is a group, this shows that $|u|$ and $|v|$ are homotopy equivalences, i.e. u and v are weak equivalences.

Rows 2 and 3 of diagrams (5.1) and (5.18) can be compared in the same fashion. For rows 4 and 5 some extra ideas are required. For example, to compare $\operatorname{hocolim}_T \mathcal{W}_T$ and $\operatorname{hocolim}_T {}^{\partial}\mathcal{W}_T$ we use the sheaf

$$\operatorname*{hocolim}_{(S,T)} \mathcal{W}_S \times \mathcal{W}_T$$

where S and T are objects of $\mathscr{K}$ with $S \cap T = \emptyset$. The disjoint union maps and substitutions $U = S \sqcup T$ induce

$$\operatorname*{hocolim}_{(S,T)} \mathcal{W}_S \times \mathcal{W}_T \longrightarrow \operatorname*{hocolim}_{U} \mathcal{W}_U\,.$$

But the map given by specialization to the coordinates,

$$\operatorname*{hocolim}_{(S,T)} \mathcal{W}_S \times \mathcal{W}_T \longrightarrow (\operatorname*{hocolim}_{S} \mathcal{W}_S) \times (\operatorname*{hocolim}_{T} \mathcal{W}_T)$$

is a weak equivalence, so that we end up with an addition law on $|\operatorname{hocolim}_S \mathcal{W}_S|$. In the same way, we can make an action (up to homotopy) of $|\operatorname{hocolim}_S \mathcal{W}_S|$ on $|\operatorname{hocolim}_S {}^{\partial}\mathcal{W}_S|$. Then a choice of an element $z \in {}^{\partial}\mathcal{W}_{\emptyset}(\star)$ leads, via the action, to a map

$$\operatorname*{hocolim}_{S} \mathcal{W}_S \times z \longrightarrow \operatorname*{hocolim}_{S} {}^{\partial}\mathcal{W}_S$$

which, by the same reasoning as before, turns out to be a weak equivalence. □

Corollary 5.42 has a variant "with boundaries" which looks as follows.

Corollary 5.47. *The fiber of the forgetful map* ${}^{\partial}\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ *over* $(V, \varrho) \in \mathcal{W}_{\mathrm{loc},T}(\star)$ *is weakly equivalent to the sheaf which takes an* X *in* $\mathscr{X}$ *to the set of all smooth graphic bundles* $q\colon M \to X$ *of tangentially* Θ*-oriented* *compact d-manifolds with collared boundary, where the bundle $\partial M \to X$ is identified with*

$$-(C \sqcup S(V^{\varrho}) \times_T S(V^{-\varrho})) \times X \longrightarrow X.$$

6. The connectivity problem

6.1. *Overview and definitions.* Throughout this section we work with the sheaves ${}^{\partial}\mathcal{W}$, ${}^{\partial}\mathcal{W}_S$ introduced in Section 5.5 (which depend on the choice of a $(d-1)$-manifold C, as specified there). But we drop the superscripts and simply write $\mathcal{W}$, $\mathcal{W}_S$. We need the following extra condition on Θ. (This is satisfied by the examples listed in 2.5, except for the case $\Theta = \pi_0\mathrm{GL} \times Y$ when Y is not path-connected.)

Assumption 6.1. The action of $\pi_0\mathrm{GL}$ on $\pi_0\Theta$ is transitive.

The previous section gave us decompositions of $\mathcal{W}$ and $\mathcal{W}_{\mathrm{loc}}$ into pieces $\mathcal{W}_S$ and $\mathcal{W}_{\mathrm{loc},S}$, respectively, and a description of the homotopy fibers of the forgetful maps

$$\mathcal{W}_S \longrightarrow \mathcal{W}_{\mathrm{loc},S}$$

as certain bundle theories; cf. Corollary 5.47. For a given S in $\mathscr{K}$, the d-manifolds involved are typically not connected. In this section we remedy this by showing that upon taking the homotopy colimit over S, we can in fact assume that the relevant d-manifolds are connected.

Definition 6.2. For X in $\mathscr{X}$ let $\mathcal{W}_{c,S}(X) \subset \mathcal{W}_S(X)$ consist of the triples (q, V, e) as in Definition 5.46, with $q\colon M \to X$ etc., such that the bundle projection $M \smallsetminus \mathrm{im}(e) \longrightarrow X$ has connected fibers.

Then $\mathcal{W}_{c,S}$ is a subsheaf of $\mathcal{W}_S$ and $|\mathcal{W}_{c,S}|$ is a union of connected components of $|\mathcal{W}_S|$. The forgetful map from $\mathcal{W}_{c,S}$ to $\mathcal{W}_{\mathrm{loc},S}$ still has the concordance lifting property. By analogy with Corollary 5.47, we have the following analysis of its fibers.

Corollary 6.3. *The fiber of the forgetful map $\mathcal{W}_{c,S} \to \mathcal{W}_{\mathrm{loc},S}$ over V in $\mathcal{W}_{\mathrm{loc},S}(\star)$ is weakly equivalent to the sheaf which takes an X in $\mathscr{X}$ to the set of all smooth graphic bundles $q\colon M \to X$ of tangentially Θ-oriented smooth compact connected d-manifolds, where the boundary of each fiber M_x is identified with*

$$-(C \sqcup S(V^{\varrho}) \times_S S(V^{-\varrho})).$$

It would therefore be nice to have a statement saying that the inclusion of $\mathrm{hocolim}_S\, \mathcal{W}_{c,S}$ in $\mathrm{hocolim}_S\, \mathcal{W}_S$ is a weak equivalence. Unfortunately such a statement is nonsensical if we insist on letting S run through the entire category $\mathscr{K}$. We have a contravariant functor $S \mapsto \mathcal{W}_S$ from $\mathscr{K}$ to the category of sheaves on $\mathscr{X}$, but we do *not* have a subfunctor $S \mapsto \mathcal{W}_{c,S}$. It is not the case that the map

$$(k,\varepsilon)^*\colon \mathcal{W}_T \to \mathcal{W}_S$$

induced by a morphism $(k,\varepsilon)\colon S \to T$ in $\mathscr{K}$ will always map the subsheaf $\mathcal{W}_{c,T}$ to the subsheaf $\mathcal{W}_{c,S}$. Let us take a more careful look at this phenomenon.

We may assume that k is an inclusion and that $T \smallsetminus S$ has exactly one element t, with label $\lambda(t) \in \{0,1,\dots,d,d+1\}$ and sign $\varepsilon(t) \in \{\pm 1\}$. Fix (q,V,e) in $\mathcal{W}_T(X)$, with $q\colon M \to X$ and let (q',V',e') be the image of (q,V,e) in $\mathcal{W}_S(X)$, with $q'\colon M' \to X$. For each $x \in X$ there is a canonical embedding

$$M_x \smallsetminus \mathrm{im}(e_x) \longrightarrow M'_x \smallsetminus \mathrm{im}(e'_x).$$

The complement of its image is identified with

$$\begin{array}{lll} D(V^{\varrho}_{(t,x)}) \times S(V^{-\varrho}_{(t,x)}) & \text{if} \quad \varepsilon(t) = +1, & \text{and} \\ S(V^{\varrho}_{(t,x)}) \times D(V^{-\varrho}_{(t,x)}) & \text{if} \quad \varepsilon(t) = -1, & \end{array}$$

where $V_{(t,x)}$ is the fiber of V over $(t,x) \in T \times X$. We have a problem when the complement is nonempty but has empty boundary, because then it will contribute an additional connected component. This happens precisely when $(\lambda(t),\varepsilon(t)) = (d+1,+1)$ and when $(\lambda(t),\varepsilon(t)) = (0,-1)$. In all other cases, there is no problem.

Now our indexing category $\mathscr{K}$ is equivalent to a product $\mathscr{K}' \times \mathscr{K}''$. The categories $\mathscr{K}'$ and $\mathscr{K}''$ can be described as full subcategories of $\mathscr{K}$; namely, $\mathscr{K}'$ is spanned by the objects S whose reference map $S \to \{0,1,2,\dots,d+1\}$ has image contained in $\{0,d+1\}$ and $\mathscr{K}''$ is spanned by the objects S whose reference map $S \to \{0,1,2,\dots,d+1\}$ has image contained in $\{1,2,\dots,d\}$.

For homotopy colimits of functors from a product category to spaces (or to sheaves on $\mathscr{X}$) there is a Fubini principle. In our case it states that

$$\underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ \mathcal{W}_T \ \simeq\ \underset{Q \text{ in } \mathscr{K}'}{\mathrm{hocolim}}\ \ \underset{S \text{ in } \mathscr{K}''}{\mathrm{hocolim}}\ \mathcal{W}_{Q \amalg S}. \tag{6.1}$$

Lemma 6.4. *For any morphism $(k,\varepsilon)\colon P \to Q$ in $\mathscr{K}'$, the commutative square*

$$\begin{array}{ccc} \underset{S \text{ in } \mathscr{K}''}{\mathrm{hocolim}}\ \mathcal{W}_{Q\amalg S} & \xrightarrow{(k,\varepsilon)^*} & \underset{S \text{ in } \mathscr{K}''}{\mathrm{hocolim}}\ \mathcal{W}_{P\amalg S} \\ \downarrow & & \downarrow \\ \underset{S \text{ in } \mathscr{K}''}{\mathrm{hocolim}}\ \mathcal{W}_{\mathrm{loc},Q\amalg S} & \xrightarrow{(k,\varepsilon)^*} & \underset{S \text{ in } \mathscr{K}''}{\mathrm{hocolim}}\ \mathcal{W}_{\mathrm{loc},P\amalg S} \end{array}$$

is homotopy cartesian (*after passage to representing spaces*).

THEOREM 6.5. *The inclusion*

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_{c,S} \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_S$$

is a weak equivalence.

Theorem 6.5 is the main result of the section. We develop a surgery method to prove it. The idea is to make nonconnected d-manifolds connected by means of multiple surgeries on embedded (thickened) 0-spheres, i.e., by replacing (fiberwise) disjoint unions by connected sums.

6.2. *Categories of multiple surgeries.* In this section we fix a compact, smooth, nonempty d-manifold M with a Θ-orientation of TM. Unless otherwise stated, $\mathbb{R}^{d+1}$ will be regarded as a Morse vector space with the standard inner product and involution $\varrho(x_1, \ldots, x_d, x_{d+1}) = (x_1, \ldots, x_d, -x_{d+1})$. We shorten $D((\mathbb{R}^{d+1})^{\varrho}) \times S((\mathbb{R}^{d+1})^{-\varrho})$ to $D^d \times S^0$. This is normally identified with what we have previously called

$$(f_V^+)^{-1}(0) \subset \mathrm{sdl}(V, \varrho)$$

in the case $V = \mathbb{R}^{d+1}$; cf. (5.8). Hence any Θ-orientation on the tangent bundle of $\mathbb{R}^{d+1}$ will induce one on the tangent bundle of $D^d \times S^0$. We refer to the discussion leading up to Proposition 5.39.

We make a slight change in the definitions of $\mathcal{W}_S$ and $\mathcal{W}_{\mathrm{loc},S}$. Namely, where Definitions 5.2 and 5.3 ask for a Morse vector bundle $\omega\colon V \to S \times X$ with a Θ-orientation on V itself, we will now just insist on a Θ-orientation on the Morse vector bundle ω^*V with base space V. This change does not affect the homotopy types of $|\mathcal{W}_S|$ and $|\mathcal{W}_{\mathrm{loc},S}|$.

Definition 6.6. Let $\mathscr{C}_M$ be the topological category defined as follows. An object consists of a finite set T, a Θ-orientation on the tangent bundle of $\mathbb{R}^{d+1} \times T$ and a smooth embedding e_T of $D^d \times S^0 \times T$ in $M \smallsetminus \partial M$ which respects the Θ-orientations and satisfies the following condition: Surgery on e_T results in a *connected* d-manifold. A morphism from (S, e_S) to (T, e_T) is an injective map $k\colon S \to T$ such that $k^*e_T = e_S$.

The category $\mathscr{C}_M$ has a natural topology: $\mathrm{ob}(\mathscr{C}_M)$ is topologized as a subspace of the disjoint union, over all T, of the spaces

$$\begin{aligned}&(\text{space of smooth embeddings } D^d \times S^0 \times T \longrightarrow M \smallsetminus \partial M)\\ &\qquad \times (\text{space of } \Theta\text{-orientations on the tangent bundle of } \mathbb{R}^d \times T).\end{aligned}$$

The total morphism set $\mathrm{mor}(\mathscr{C}_M)$ is topologized as a subset of $\mathrm{ob}(\mathscr{C}_M) \times \mathrm{ob}(\mathscr{C}_M)$ via the map (source,target).

PROPOSITION 6.7. *The space $B\mathscr{C}_M$ is contractible.*

The proof requires a lemma.

LEMMA 6.8. *Let* $\sigma: N \to X$ *be a submersion of smooth manifolds without boundary, with* $\dim(N) > \dim(X)$. *Suppose that for each* $x \in X$ *there exists a contractible open neighborhood* W *of* x *in* X, *a finite set* Q *and a map* $Q \times W \to N$ *over* X *inducing a surjection from* Q *to* $\pi_0(N_y)$ *for every* $y \in W$. *Then there exist a locally finite covering of* X *by contractible open sets* W_j, *where* $j \in J$, *and finite sets* Q_j, *and a smooth embedding*

$$a\colon \coprod_j Q_j \times W_j \longrightarrow N$$

over X, *such that for each* $j \in J$ *and* $x \in W_j$, *the restriction of* a *to* $Q_j \times W_j$ *induces a surjection* $Q_j \to \pi_0(N_x)$.

Example 6.9. The submersion $\mathbb{R}^2 \smallsetminus (0,0) \longrightarrow \mathbb{R}$; $(x,y) \mapsto x$ satisfies the hypothesis of Lemma 6.8. The submersion $\mathbb{R} \smallsetminus 0 \to \mathbb{R}$; $x \mapsto x$ does not, and neither does the projection from $(\mathbb{R} \times \{0,1\}) \smallsetminus (0,0)$ to $\mathbb{R}$.

Proof of Lemma 6.8. Note first that the statement is not completely trivial. Using the hypothesis, we could start with a locally finite covering of X by contractible open sets W_j, and choose finite sets Q_j and maps $a_j\colon Q_j \times W_j \to N$ over X inducing surjections $Q_j \to \pi_0(N_y)$ for every $y \in W_j$. This would give us a map

$$a\colon \coprod_j Q_j \times W_j \longrightarrow N$$

which is an immersion. Unfortunately there is no guarantee that it is an embedding. To solve this problem we will partition a "large", dense open subset U of N into "levels" indexed by the real numbers, and arrange that a maps distinct connected components of $\coprod Q_j \times W_j$ to distinct levels of U. Then a is an embedding.

The jet transversality theorem, applied to sections of the vertical tangent bundle of N, implies that we can find a $k \gg 0$ and a smooth $f\colon N \to \mathbb{R}$ such that the fiberwise k-jet prolongation $j^k_\sigma f\colon N \to J^k_\sigma(N,\mathbb{R})$ is nowhere 0. Let $U \subset N$ consist of all $z \in N$ such that $f|N_{\sigma(z)}$ is regular at z. Then U is open in N and $U_x := U \cap N_x$ is dense in N_x, for each $x \in X$. Hence the inclusions $U_x \to N_x$ induce surjections $\pi_0(U_x) \to \pi_0(N_x)$. The hypotheses on σ now give us a covering of X by contractible open subsets W_j, and for each W_j a finite set Q_j and a map $a_j\colon Q_j \times W_j \to U$ over X such that the induced composite map $Q_j \to \pi_0(U_x) \to \pi_0(N_x)$ is onto for every $x \in W_j$. We can assume that the W_j are the open stars of the vertices in a sufficiently fine triangulation of X, in which case the covering is locally finite. But in addition we can easily arrange that fa_j is constant on $q \times W_j$ for each $q \in Q_j$, and that the resulting map $\coprod_j Q_j \to \mathbb{R}$ is injective. Then the map a which equals a_j on $Q_j \times W_j$ satisfies all our requirements. $\square$

In the proof of Theorem 6.5, we will use a sheaf version $\mathcal{C}_M$ of $\mathscr{C}_M$. For connected X in $\mathscr{X}$ let $\mathcal{C}_M(X)$ be the category whose objects are the pairs (T, e_T) where T is a finite set together with a Θ-orientation on the tangent bundle of $\mathbb{R}^{d+1} \times T$, and

$$e_T \colon D^d \times S^0 \times T \times X \quad \longrightarrow \quad (M \smallsetminus \partial M) \times X$$

is a smooth embedding over X, respecting the tangential Θ-orientations and subject to the condition that fiberwise surgery on e_T results in a bundle of connected manifolds. A morphism from (S, e_S) to (T, e_T) is an injective map $k \colon S \to T$ such that $k^* e_T = e_S$.

Since $\mathrm{ob}(\mathcal{C}_M(\Delta^k_e)) = C^\infty(\Delta^k_e, \mathrm{ob}(\mathscr{C}_M))$ as sets, one gets a functor of topological categories $|\mathcal{C}_M| \to \mathscr{C}_M$ which induces a degreewise homotopy equivalence of the nerves and therefore a homotopy equivalence $B|\mathcal{C}_M^{\mathrm{op}}| \cong B|\mathcal{C}_M| \to B\mathscr{C}_M$. (Here it is best to define $B\mathscr{C}_M$ as the fat realization [39] of the nerve of $\mathscr{C}_M$, ignoring the degeneracy operators.)

Proof of Proposition 6.7. We show that $\beta\mathcal{C}_M^{\mathrm{op}}$ is weakly equivalent to the terminal sheaf taking every X in $\mathscr{X}$ to a singleton. By Proposition 2.18, this reduces to the following

> *Claim.* Let X in $\mathscr{X}$ be given with a closed subset A and a germ s in $\mathrm{colim}_U\, \beta\mathcal{C}_M^{\mathrm{op}}(U)$, where U ranges over the neighborhoods of A in X. Then s extends to an element of $\beta\mathcal{C}_M^{\mathrm{op}}(X)$.

To verify this, choose an open neighborhood U of A in X such that the germ s can be represented by some $s_0 \in \beta\mathcal{C}_M^{\mathrm{op}}(U)$. The information contained in s_0 includes a locally finite covering of U by open subsets U_j for $j \in J$. (Making U smaller if necessary, we can assume that this is locally finite in the strong sense that every $x \in X$ has a neighborhood which meets only finitely many U_j.) It also includes a choice of object $\psi_{RR} \in \mathrm{ob}(\mathcal{C}_M(U_R))$ for each finite nonempty subset R of J. (There are also morphisms $\psi_{RS} \in \mathrm{mor}(\mathcal{C}_M(U_S))$, but they are of course determined by their sources $\psi_{RR}|U_S$ and targets ψ_{SS}.)

Next, choose an open $X_0 \subset X$ such that $U \cup X_0 = X$ and the closure of X_0 in X avoids A. Let N be the open subset of $(M \smallsetminus \partial M) \times X_0$ obtained by removing from $(M \smallsetminus \partial M) \times X_0$ the closures of the embedded disk bundles determined by the various $\varphi_{RR}|U_R \cap X_0$. By making U and X_0 and the U_j smaller if necessary, but taking care that the U_j remain the same near A, we can arrange that the projection $N \to X_0$ satisfies the hypothesis of Lemma 6.8.

Thus there exists a locally finite covering of X_0 by contractible open sets U'_j, and finite sets Q_j and an embedding a of $\coprod_j Q_j \times U'_j$ in N, over X_0 , such that a induces surjections $Q_j \to \pi_0(N_x)$ for each j and $x \in U'_j$. (Again, making

X_0 smaller if necessary, we can assume that this is locally finite in the strong sense that every $x \in X$ has a neighborhood which meets only finitely many U'_j.) We can also choose a smooth embedding b of $\coprod_j Q_j \times U'_j$ in N, over X_0, inducing constant maps $Q_j \to \pi_0(N_x)$ for each j and all $x \in U'_j$, and such that $\mathrm{im}(a) \cap \mathrm{im}(b) = \emptyset$. (For example, the distinct sheets of b restricted to $Q_j \times U'_j$ can be chosen very close to a selected sheet of a.) Since the U'_j are contractible, the normal bundles of a and b can be trivialized (as d-dimensional vector bundles), and so the "union" of a and b extends to a smooth and fiberwise Θ-orientation preserving embedding

$$c\colon D^d \times S^0 \times \textstyle\coprod_j (Q_j \times U'_j) \quad\longrightarrow\quad N$$

over X_0, with suitably chosen Θ-orientations on the vertical tangent bundles of the projections $\mathbb{R}^{d+1} \times Q_j \times U'_j \longrightarrow Q_j \times U'_j$. (This requires assumption 6.1.) For each j with nonempty U'_j, the restriction of c to the summand

$$D^d \times S^0 \times Q_j \times U'_j$$

is an object φ_{jj} of $\mathcal{C}_M(U'_j)$. Assuming that J is uncountable, we can arrange that U'_j is empty whenever U_j is nonempty.

We are now ready to define an explicit element in $\beta\mathcal{C}_M^{\mathrm{op}}(X)$ which extends the germ s. Let $Y_j = U_j$ if U_j is nonempty, $Y_j = U'_j$ if U'_j is nonempty, and $Y_j = \emptyset$ for all other $j \in J$. Then the Y_j form a locally finite open covering of X. For finite $R \subset J$ with nonempty Y_R, we can write $Y_R = U_S \cap U'_T$ for disjoint subsets S, T of R with $S \cup T = R$. Let $\varphi_{RR} \in \mathrm{ob}(\mathcal{C}_M(Y_R))$ be the coproduct (which exists by construction) of $\psi_{SS}|Y_R$ and the $\varphi_{jj}|Y_R$ for $j \in T$. The covering $j \mapsto Y_j$ together with the data φ_{RR} for finite nonempty $R \subset J$ is an element in $\beta\mathcal{C}_M^{\mathrm{op}}(X)$ which extends the germ s. □

We call a diagram $S \to T \leftarrow U$ in $\mathscr{K}''$, given by morphisms $(k_1, \varepsilon_1)\colon S \to T$ and $(k_2, \varepsilon_2)\colon U \to T$, *special* if $k_2(U)$ contains $k_1(S)$, all elements of $T \smallsetminus k_1(S)$ have label $1 \in \{0, 1, 2, \ldots, d+1\}$, and $\varepsilon_1 \equiv +1$, $\varepsilon_2 \equiv -1$. In that situation we also define $\mathcal{W}_{c,\,U\to T}$ by means of the pullback diagram

$$\begin{tikzcd}
\mathcal{W}_{c,\,U\to T} \arrow[r] \arrow[d] & \mathcal{W}_{c,U} \arrow[d, "\text{inclusion}"] \\
\mathcal{W}_T \arrow[r, "(k_2,\varepsilon_2)^*"] & \mathcal{W}_U\,.
\end{tikzcd}$$

The special diagrams $S \to T \leftarrow U$ with a fixed S are the objects of a category $\mathscr{D}_S$ where the morphisms are commutative diagrams in $\mathscr{K}''$ of the form

(6.2)
$$\begin{tikzcd}
S \arrow[r] \arrow[d, "="] & T \arrow[d] & U \arrow[l] \arrow[d] \\
S \arrow[r] & T' & U' \arrow[l]
\end{tikzcd}$$

with special rows. In such a diagram, every element z of T' which is not in the image of T must be in the image of U'. Indeed, writing $(k, \varepsilon) \colon T \to T'$ for the morphism in the middle column, we have $\varepsilon(z) = +1$ by the commutativity of the left-hand square, and $\varepsilon(z) = +1$ implies that z is in the image of U' by the commutativity of the right-hand square. Therefore the rule taking a special diagram $S \to T \leftarrow U$ to $\mathcal{W}_{c,\, U\to T}$ is a contravariant functor on $\mathscr{D}_S$. There is also a natural transformation from that functor on $\mathscr{D}_S$ to the constant functor with value $\mathcal{W}_S$, determined by the composition

$$\mathcal{W}_{c,\, U\to T} \longrightarrow \mathcal{W}_T \longrightarrow \mathcal{W}_S$$

for $S \to T \leftarrow U$ in $\mathscr{D}_S$.

LEMMA 6.10. *This natural transformation induces a homotopy equivalence*

$$\underset{S\to T\leftarrow U \text{ in } \mathscr{D}_S}{\operatorname{hocolim}} \ |\mathcal{W}_{c,\, U\to T}| \ \longrightarrow \ |\mathcal{W}_S| \ .$$

Proof. We will proceed by showing that all homotopy fibers of the map are contractible, and for that we use Lemma 6.11 below. This means that we must select a point in $|\mathcal{W}_S|$, corresponding to a certain d-manifold N with an embedding e of a disjoint union of thickened spheres, and we must then show that

$$(*) \qquad \underset{S\to T\leftarrow U \text{ in } \mathscr{D}_S}{\operatorname{hocolim}} \quad \operatorname{hofiber}_{(N,e)}\big[\,|\mathcal{W}_{c,\, U\to T}| \to |\mathcal{W}_S|\,\big]$$

is contractible. Let $M = N \smallsetminus \operatorname{int}(\operatorname{im}(e))$. For a fixed $S \to T \leftarrow U$ in $\mathscr{D}_S$, it is easy to produce a chain of natural homotopy equivalences from

$$\operatorname{hofiber}_{(N,e)}\big[\,|\mathcal{W}_{c,\, U\to T}| \to |\mathcal{W}_S|\big]$$

to the space of morphisms in $\mathscr{C}_M$ whose underlying set map is the inclusion $T_0 \to T_1$, where T_1 is the complement of the image of S in T and T_0 is the complement of the image of U in T. (There are two kinds of connectedness conditions to be compared at this point. One kind requires that certain surgeries on M produce a connected manifold. The other requires that certain surgeries on $M \smallsetminus \operatorname{int}(K)$ produce a connected manifold, where $K \subset M$ is a disjoint union of copies of $D^d \times S^0$. But clearly, removing $\operatorname{int}(K)$ does not influence the π_0 in question.) The homotopy colimit $(*)$ is therefore homotopy equivalent to

$$\underset{v\colon T_0\to T_1}{\operatorname{hocolim}} \ \ (\text{space of morphisms in } \mathscr{C}_M \text{ with underlying set map } v).$$

This homotopy colimit is the classifying space of the *edgewise subdivision* of $\mathscr{C}_M$: given the diagram (6.2) and a morphism in $\mathscr{C}_M$ with underlying set map

$v\colon T_0 \to T_1$, we get a diagram in $\mathscr{C}_M$ with underlying set maps

$$\begin{tikzcd} T_0 \arrow[r, "v"] & T_1 \arrow[d] \\ T_0' \arrow[u] \arrow[r, "v'"] & T_1' \end{tikzcd}$$

as explained in the paragraph following (6.2). In general, the edgewise subdivision es$\mathscr{A}$ of a category $\mathscr{A}$ has ob(es$\mathscr{A}$) = mor($\mathscr{A}$), and morphisms from $v = (T_0 \to T_1)$ to $v' = (T_0' \to T_1')$ are diagrams like the one above. The nerve of es$\mathscr{A}$ is, by [13, Lm.2.4], isomorphic as a simplicial set to the edgewise subdivision of the nerve of $\mathscr{A}$, and by [40] this implies that the realizations are homeomorphic. It follows that the classifying space of the edgewise subdivision of $\mathscr{C}_M$ is homotopy equivalent to the classifying space of $\mathscr{C}_M$, and therefore contractible. □

Lemma 6.11. *Let $\mathscr{A}$ be a (small) category, $\mathcal{F}$ a functor from $\mathscr{A}$ to spaces, p a natural transformation from $\mathcal{F}$ to a constant functor with value B (a* CW-*space), and $b \in B$. Then there is a chain of weak homotopy equivalences*

$$\mathrm{hofiber}_b\left[\mathrm{hocolim}_a\, \mathcal{F}(a) \xrightarrow{p_*} B\right] \quad \simeq \quad \mathrm{hocolim}_a\left(\mathrm{hofiber}_b[\mathcal{F}(a) \to B]\right).$$

Proof. By the homotopy invariance property of homotopy direct limits, we may assume that $p_a\colon \mathcal{F}(a) \to B$ is a (Serre) fibration for every a in $\mathscr{A}$. In that situation the map $\mathrm{hocolim}_a\, \mathcal{F}(a) \longrightarrow B$ is a quasifibration and its fiber over b is

$$\mathrm{hocolim}_a\left(\mathrm{fiber}_b[\mathcal{F}(a) \to B]\right) \;\simeq\; \mathrm{hocolim}_a\left(\mathrm{hofiber}_b[\mathcal{F}(a) \to B]\right). \qquad \square$$

Let $\mathscr{D}$ be the category of *all* special diagrams $S \to T \leftarrow U$ in $\mathscr{K}''$, so that a morphism in $\mathscr{D}$ is a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] & U \arrow[l] \arrow[d] \\ S' \arrow[r] & T' & U' \arrow[l] \end{tikzcd}$$

in $\mathscr{K}''$ with special rows. As before, the rule taking an object $S \to T \leftarrow U$ of $\mathscr{D}$ to $|\mathcal{W}_{c,\, U\to T}|$ is a contravariant functor. There is an embedding

$$\mathrm{hocolim}_S|\mathscr{D}_S| \longrightarrow |\mathscr{D}|. \tag{6.3}$$

Indeed, the left-hand side can be regarded as the geometric realization of a simplicial set whose n-simplices are pairs of diagrams in $\mathscr{K}''$ of the form

$$S_0 \to S_1 \to \cdots \to S_n \quad ; \quad S_n \to T \leftarrow U$$

where the second diagram is special. Such a diagram can also be viewed as a string of n composable morphisms in $\mathscr{D}$ by replicating T and U.

Proof of Theorem 6.5. Let

$$X = \operatorname*{hocolim}_{S\to T\leftarrow U \text{ in } \mathscr{D}} |\mathcal{W}_{c,\, U\to T}|\,, \quad X_S = \operatorname*{hocolim}_{S\to T\leftarrow U \text{ in } \mathscr{D}_S} |\mathcal{W}_{c,\, U\to T}|.$$

There are maps

$$[0,2]\times \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} X_S \xrightarrow{\ v\ } [0,2]\times X \xrightarrow{\ g\ } \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |\mathcal{W}_S|$$

where v is induced by (6.3) and g is induced by the three functors f_0, f_1, f_2 from $\mathscr{D}$ to $\mathscr{K}''$ defined by $(S\to T\leftarrow U)\mapsto S,T,U$ respectively, and the two obvious natural transformations $f_0\to f_1$, $f_2\to f_1$. (Each f_i induces a map from X to $\mathrm{hocolim}_S|\mathcal{W}_S|$ and the two natural transformations induce two homotopies, which we concatenate to obtain a single map from $[0,2]\times X$ to $\mathrm{hocolim}_S|\mathcal{W}_S|$.)

Let $h = gv$ and write h_t for the restriction of gv to $t\times \mathrm{hocolim}_S X_S$. It follows from Lemma 6.10 and the homotopy invariance property of homotopy colimits that h_0 is a homotopy equivalence. By construction, h_2 lands in the subspace $\mathrm{hocolim}_S|\mathcal{W}_{c,S}|$ of $\mathrm{hocolim}_S|\mathcal{W}_S|$. Hence $h_2h_0^{-1}$ is a homotopy class of maps from $\mathrm{hocolim}_S|\mathcal{W}_S|$ to $\mathrm{hocolim}_S|\mathcal{W}_{c,S}|$ which is right inverse to the inclusion. But it is also left inverse to the inclusion, because X contains a copy of $\mathrm{hocolim}_S|\mathcal{W}_{c,S}|$ on which both h_2 and h_0 are the identity. □

6.3. *Annihiliation of d-spheres.* The goal is to prove Lemma 6.4. Most of the proof is based on some elementary product decompositions.

Lemma 6.12. *Let $T = T_1\cup T_2$ be a disjoint union, where T_1 is an object of $\mathscr{K}'$ and T_2 is an object of $\mathscr{K}$. There are weak equivalences*

$$\mathcal{W}_T \longrightarrow \mathcal{W}_{\mathrm{loc},T_1}\times \mathcal{W}_{T_2}\,, \qquad \mathcal{W}_{\mathrm{loc},T} \longrightarrow \mathcal{W}_{\mathrm{loc},T_1}\times \mathcal{W}_{\mathrm{loc},T_2},$$

natural in T_2 for fixed T_1.

Proof. The second map is induced by the inclusions $T_1\to T$ and $T_2\to T$, and is obviously a weak equivalence.

The first coordinate of the first map is again induced by the inclusion $T_1\to T$. The second coordinate, $\mathcal{W}_T\longrightarrow\mathcal{W}_{T_2}$, is defined as follows. Let (q,V,e) be an element of $\mathcal{W}_T(X)$ as in Definition 5.3, with $q\colon M\to X$. For $a\in T_1$, the bundle

$$D(V_a^{\varrho})\times_{X_a} S(V_a^{-\varrho})$$

(where $X_a = a\times X$ and $V_a = V|X_a$) is either empty or a bundle of d-spheres. In any case it has empty boundary and its image under e is a union of connected components of M. Let M' be obtained from M by deleting these components, for all $a\in T_1$. Let V' be the restriction of V to $T_2\times X$ and let e' be the restriction of e to

$$\coprod_{b\in T_2} D(V_b^{\varrho}) \times_{X_b} S(V_b^{-\varrho})\,.$$

Then $(q',V',e') \in \mathcal{W}_{T_2}(X)$. This determines the map $\mathcal{W}_T \longrightarrow \mathcal{W}_{T_2}$. Again it should be clear that the resulting map

$$\mathcal{W}_T \longrightarrow \mathcal{W}_{\mathrm{loc},T_1} \times \mathcal{W}_{T_2}$$

is a weak equivalence: it is easy to write down an inverse for the induced map on homotopy groups. □

Proof of Lemma 6.4. Applying Lemma 6.12 we can rewrite the commutative diagram in Lemma 6.4 in the form

$$\begin{array}{ccc}
|\mathcal{W}_{\mathrm{loc},Q}| \times |\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_S| & \xrightarrow{(k,\varepsilon)^*} & |\mathcal{W}_{\mathrm{loc},P}| \times |\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_S| \\
\Big\downarrow{\scriptstyle \mathrm{id}\times\ell} & & \Big\downarrow{\scriptstyle \mathrm{id}\times\ell} \\
|\mathcal{W}_{\mathrm{loc},Q}| \times |\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_{\mathrm{loc},S}| & \xrightarrow{(k,\varepsilon)^*} & |\mathcal{W}_{\mathrm{loc},P}| \times |\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_{\mathrm{loc},S}|
\end{array}$$

where $\ell\colon \mathrm{hocolim}_S\,\mathcal{W}_S \longrightarrow \mathrm{hocolim}_S\,\mathcal{W}_{\mathrm{loc},S}$ is the forgetful map. For y in $\mathcal{W}_{\mathrm{loc},Q}(\star)$ and $z \in \mathrm{hocolim}_S\,\mathcal{W}_S(\star)$, the homotopy fiber of the left-hand vertical arrow over (y,z) is therefore identified with $\mathrm{hofiber}_z(\ell)$ and the homotopy fiber of the right-hand vertical arrow over the image point $((k,\varepsilon)^*y,z)$ is also identified with $\mathrm{hofiber}_z(\ell)$. However, with these identifications the map

$$u\colon \mathrm{hofiber}_z(\ell) \longrightarrow \mathrm{hofiber}_z(\ell)$$

induced by the horizontal arrows in the diagram is not always the identity. To understand what it is, we can assume that $Q \smallsetminus P$ has exactly one element a. Associated with this we have a label $\lambda(a) \in \{0, d+1\}$ and a value $\varepsilon(a)$ in $\{-1,+1\}$. By inspection, if $(\lambda(a),\varepsilon(a))$ is $(0,+1)$ or $(d+1,-1)$, then the map u is the identity. To describe what happens in the remaining cases, we note that by choosing y we have also selected an element

$$(p,W,g) \in \mathcal{W}_{\mathrm{loc},\{a\}}(\star)$$

where W is a vector space with inner product. We identify W with $\mathbb{R}^{d+1}$, so the map u is given by disjoint union with $S(W) = S^d$, assuming that $(\lambda(a),\varepsilon(a))$ is $(d+1,+1)$ or $(0,-1)$. More precisely, for each S in $\mathscr{K}''$ and X in $\mathscr{X}$, we have a map

$$\bar{u}\colon \mathcal{W}_S(X) \to \mathcal{W}_S(X)$$

given by $(q,V,e) \mapsto (q^\sharp,V,e)$ where $q\colon M \to X$ is a bundle of d-manifolds etc., and $q^\sharp$ is obtained from q by disjoint union with a trivial sphere bundle $S^d \times X \to X$. This is natural in the variables X and S. It covers the identity map of $\mathcal{W}_{\mathrm{loc},S}(X)$ and so induces u above. Hence it only remains to show that $\bar{u}$ is a weak equivalence.

LEMMA 6.13. *The map*

$$\bar{u}\colon \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_S \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} \mathcal{W}_S$$

given by disjoint union of all d-manifolds in sight with S^d is a weak equivalence.

Proof. We reason as in Section 5.7. This will require two variants of $\mathcal{W}_S$ as in Definition 5.3, one where we use $-C$ as the prescribed boundary and another where we use $C \sqcup -C$, in other words, the boundary of $C \times [0,1]$. To distinguish these, we write ${}^{\partial}\mathcal{W}_S$ for the first and ${}^{\partial\partial}\mathcal{W}_S$ for the second.

Concatenation defines a map from ${}^{\partial\partial}\mathcal{W}_S \times {}^{\partial\partial}\mathcal{W}_T$ to ${}^{\partial\partial}\mathcal{W}_{S\sqcup T}$ and another map from ${}^{\partial\partial}\mathcal{W}_S \times {}^{\partial}\mathcal{W}_T$ to ${}^{\partial}\mathcal{W}_{S\sqcup T}$. Hence

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |{}^{\partial\partial}\mathcal{W}_S|$$

becomes a homotopy monoid. Its homotopy unit is the element $C \times [0,1]$ in ${}^{\partial\partial}\mathcal{W}_\emptyset(\star)$. This homotopy monoid acts on

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |{}^{\partial}\mathcal{W}_S|.$$

The map $\bar{u}$ is given by translation with the single element z of

$$|{}^{\partial\partial}\mathcal{W}_\emptyset| \subset \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |{}^{\partial\partial}\mathcal{W}_S|$$

defined by $S^d \sqcup (C \times [0,1])$. It is therefore enough to show that z is in the connected component of the homotopy unit, defined by $C \times [0,1]$. This amounts to saying that $S^d \sqcup (C \times [0,1])$ can be transformed into $C \times [0,1]$ by elementary surgeries of index $1, 2, \ldots, d$ only. In fact a single surgery of index 1, that is, a surgery on a thickened 0-sphere in $S^d \sqcup (C \times [0,1])$, is enough. (Let one component of the thickened 0-sphere be in S^d and the other in $C \times [0,1]$. Here at last we are using the assumption that $C \neq \emptyset$.) □

7. Stabilization and proof of the main theorem

7.1. *Stabilizing the decomposition.*

Conventions. Throughout this section we assume $d = 2$ and $\Theta = \pi_0\mathrm{GL}$ with the translation action of GL, so that Θ-orientations are ordinary orientations. We continue to write $\mathcal{W}$ and $\mathcal{W}_S$ for ${}^{\partial}\mathcal{W}$ and ${}^{\partial}\mathcal{W}_S$, respectively. The fixed boundary C is $S^1 \sqcup -S^1$; cf. Section 5.7.

In Section 6, we modified the homotopy colimit decomposition of $|\mathcal{W}|$ obtained in Section 5 in order to banish nonconnected d-manifolds from the picture, as far as possible. Here we make a second modification to our homotopy colimit decomposition which, roughly speaking, ensures that all surfaces in sight are of large genus, in addition to being connected. We achieve this by repeatedly concatenating with a standard surface of genus 1, with boundary $S^1 \sqcup -S^1$. This standard surface can be viewed as an element $z \in \mathcal{W}_{c,\emptyset}(\star)$.

For every X in $\mathscr{X}$, the unique map $X \to \star$ induces $\mathcal{W}_{c,\emptyset}(\star) \to \mathcal{W}_{c,\emptyset}(X)$ and so allows us to think of z as an element of $\mathcal{W}_{c,\emptyset}(X)$. For S in $\mathscr{K}$ define $z^{-1}\mathcal{W}_S$ and $z^{-1}\mathcal{W}_{c,S}$ as the colimits, in the category of sheaves on $\mathscr{X}$, of the diagrams

$$\mathcal{W}_S \xrightarrow{z\cdot} \mathcal{W}_S \xrightarrow{z\cdot} \mathcal{W}_S \xrightarrow{z\cdot} \mathcal{W}_S \xrightarrow{z\cdot} \cdots,$$
$$\mathcal{W}_{c,S} \xrightarrow{z\cdot} \mathcal{W}_{c,S} \xrightarrow{z\cdot} \mathcal{W}_{c,S} \xrightarrow{z\cdot} \mathcal{W}_{c,S} \xrightarrow{z\cdot} \cdots,$$

respectively. The arrows labelled $z\cdot$ are given by concatenation with z. These colimits are obtained by sheafifying the naive colimits, which are presheaves. (The categorical colimit $\mathcal{F}_\infty$ of a system of sheaves

$$\mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to \cdots$$

on $\mathscr{X}$ can be defined explicitly by $\mathcal{F}_\infty(X) = \lim_U(\operatorname{colim}_i \mathcal{F}_i(U))$, for X in $\mathscr{X}$, where U runs through the open subsets of X which have compact closure in X.) The sheafification process does not alter the values on compact objects of $\mathscr{X}$, such as spheres. Hence the representing spaces of these colimits are homotopy equivalent to the colimits of the individual representing spaces:

$$|z^{-1}\mathcal{W}_S| \simeq z^{-1}|\mathcal{W}_S|, \qquad |z^{-1}\mathcal{W}_{c,S}| \simeq z^{-1}|\mathcal{W}_{c,S}|.$$

For an object T in $\mathscr{K}''$, Corollary 6.3 implies that the homotopy fiber of the localization map $|\mathcal{W}_{c,T}| \longrightarrow |\mathcal{W}_{\mathrm{loc},T}|$ over any base point is homotopy equivalent to $\coprod_g B\Gamma_{g,\,2+2|T|}$. The stabilization process replaces the disjoint union with $\mathbb{Z} \times B\Gamma_{\infty,\,2+2|T|}$ and so we have

LEMMA 7.1. *For T in $\mathscr{K}''$, any homotopy fiber of $|z^{-1}\mathcal{W}_{c,T}| \longrightarrow |\mathcal{W}_{\mathrm{loc},T}|$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma_{\infty,\,2+2|T|}$.* □

The stabilized version of Lemma 6.4 is that the commutative diagram

$$\begin{array}{ccc} \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_{Q\amalg S}| & \xrightarrow{(k,\varepsilon)^*} & \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_{P\amalg S}| \\ \downarrow & & \downarrow \\ \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |\mathcal{W}_{\mathrm{loc},Q\amalg S}| & \xrightarrow{(k,\varepsilon)^*} & \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |\mathcal{W}_{\mathrm{loc},P\amalg S}| \end{array} \tag{7.1}$$

is homotopy cartesian, for any morphism $(k,\varepsilon)\colon P \to Q$ in $\mathscr{K}'$. Stabilizing Theorem 6.5 gives the homotopy equivalence

$$\operatorname*{hocolim}_{T \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_{c,\,T}| \longrightarrow \operatorname*{hocolim}_{T \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_T|. \tag{7.2}$$

Finally we shall need

Lemma 7.2.

$$|\mathcal{W}| \simeq |z^{-1}\mathcal{W}| \simeq \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} |z^{-1}\mathcal{W}_T| \simeq \operatorname*{hocolim}_{Q \text{ in } \mathscr{K}'} \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_{Q\amalg S}|.$$

Proof. The space $|\mathcal{W}|$ is group complete by Theorem 1.2 and Theorem 1.4 which together imply that $|\mathcal{W}|$ is an infinite loop space. Hence the inclusion of $|\mathcal{W}|$ in $z^{-1}|\mathcal{W}| \simeq |z^{-1}\mathcal{W}|$ is a homotopy equivalence. The second homotopy equivalence in the chain follows from $|z^{-1}\mathcal{W}_T| \simeq z^{-1}|\mathcal{W}_T|$ and

$$\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} z^{-1}|\mathcal{W}_T| \simeq z^{-1}\big(\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} |\mathcal{W}_T|\big).$$

The third equivalence is the Fubini principle for homotopy colimits; cf. (6.1).

7.2. *The Harer-Ivanov stability theorem.*

Lemma 7.3. *The canonical map from $\mathbb{Z} \times B\Gamma_{\infty,2}$ to the homotopy fiber (over the base point) of the forgetful map*

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_{c,S}| \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |\mathcal{W}_{\mathrm{loc},S}|$$

induces an isomorphism in homology with integer coefficients.

Proof. For the object $S = \emptyset$ of $\mathscr{K}''$, we have $|z^{-1}\mathcal{W}_{c,S}| \simeq \mathbb{Z} \times B\Gamma_{\infty,2}$ and $|\mathcal{W}_{\mathrm{loc},S}| = \star$. This gives a canonical map from $\mathbb{Z} \times B\Gamma_{\infty,2}$ to the homotopy fiber of

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |z^{-1}\mathcal{W}_{c,S}| \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}''} |\mathcal{W}_{\mathrm{loc},S}|.$$

We now check that the hypothesis of Corollary B.3 is satisfied. Let $(k,\varepsilon)\colon S \to T$ be a morphism in $\mathscr{K}''$. We have to verify that, in the commutative square of spaces

$$\begin{tikzcd}
|z^{-1}\mathcal{W}_{c,T}| \arrow[r] \arrow[d, "(k,\varepsilon)^*"] & |\mathcal{W}_{\mathrm{loc},T}| \arrow[d, "(k,\varepsilon)^*"] \\
|z^{-1}\mathcal{W}_{c,S}| \arrow[r] & |\mathcal{W}_{\mathrm{loc},S}|,
\end{tikzcd}$$

the induced map from any of the homotopy fibers in the upper row to the corresponding homotopy fiber in the lower row induces an isomorphism in homology. The homotopy fibers in question are related by a map

$$\mathbb{Z} \times B\Gamma_{\infty,\,2+2|T|} \longrightarrow \mathbb{Z} \times B\Gamma_{\infty,\,2+2|S|}$$

given geometrically by attaching cylinders $D^1 \times S^1$ or double disks $D^2 \times S^0$ to those pairs of boundary circles which correspond to elements of $T \smallsetminus k(S)$.

This map is an integral homology equivalence by the Harer-Ivanov stability theorem. Apply Corollary B.3. $\square$

COROLLARY 7.4. *The canonical map from* $\mathbb{Z} \times B\Gamma_{\infty,2}$ *to the homotopy fiber* (*over the base point*) *of the forgetful map*

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |z^{-1}\mathcal{W}_S| \quad\longrightarrow\quad \operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |\mathcal{W}_{\mathrm{loc},S}|$$

induces an isomorphism in homology with integer coefficients.

Proof. Use the homotopy cartesian diagram (7.2), the homotopy equivalence (7.1) and the last homotopy equivalence of Lemma 7.2. $\square$

Proof of Theorem 1.5. By Lemma 7.2 and diagram 5.1, we have

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |z^{-1}\mathcal{W}_S| \;\simeq\; |\mathcal{W}|, \qquad\qquad \operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |\mathcal{W}_{\mathrm{loc},S}| \;\simeq\; |\mathcal{W}_{\mathrm{loc}}|.$$

Therefore Corollary 7.4 implies that the homotopy fiber of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ receives a map from $\mathbb{Z} \times B\Gamma_{\infty,2}$ which induces an isomorphism in integer homology. But $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ are infinite loop spaces by Theorem 1.2, and the map $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ is an infinite loop map. Hence its homotopy fiber is an infinite loop space, and each of its components has an abelian fundamental group. Each of these fundamental groups is therefore isomorphic to $H_1(B\Gamma_{\infty,2};\mathbb{Z}) = 0$. Summing up, we see that all connected components of the homotopy fiber in question are simply connected, and the homotopy fiber is therefore $\mathbb{Z} \times B\Gamma^+_{\infty,2}$. $\square$

A. More about sheaves

A.1. *Concordance and the representing space.* Let $\mathcal{F}$ be a sheaf on $\mathscr{X}$. We shall construct a natural transformation $\vartheta\colon [X, |\mathcal{F}|\,] \longrightarrow \mathcal{F}[X]$, and an inverse $\xi\colon \mathcal{F}[X] \to [X, |\mathcal{F}|\,]$ for ϑ.

We start with the construction of ξ. Fix X in $\mathscr{X}$ and an element u in $\mathcal{F}(X)$. Choose a smooth triangulation of X, with vertex set $\mathbb{T}$ which we assume equipped with a total ordering. Suppose that $S \subset \mathbb{T}$ is a distinguished subset (the vertex set of a simplex in the triangulation). Let

$$\begin{aligned} \Delta_e(S) &= \{w\colon S \to \mathbb{R} \mid \Sigma_s w(s) = 1\} \\ \Delta(S) &= \{w \in \Delta_e(S) \mid w \geq 0\}. \end{aligned}$$

The triangulation gives us characteristic embeddings $c_S\colon \Delta(S) \to X$, one for each distinguished $S \subset \mathbb{T}$. By induction on S, we can choose smooth embeddings

$$c_{e,S}\colon \Delta_e(S) \to X\ ,$$

extending the c_S and compatible with the face structure in the sense that if S is distinguished and $R \subset S$, then

(i) $c_{e,S}$ agrees with $c_{e,R}$ on $\Delta_e(R) \subset \Delta_e(S)$.

These choices can be made in such a way that there is a smooth homotopy $(h_t\colon X \to X)_{t\in[0,1]}$, with $h_0 = \mathrm{id}$ and

(ii) h_t maps each simplex $c_S(\Delta(S))$ to itself,

(iii) h_t maps each extended simplex $c_{e,S}(\Delta_e(S))$ to itself,

(iv) each simplex $c_S(\Delta(S))$ has an open neighborhood V_S in X with

$$h_1(V_S) \subset c_S(\Delta(S)).$$

A triangulation of X with totally ordered vertex set $\mathbb{T}$ and a choice of homotopy (h_t) and embeddings $c_{e,S}$ satisfying (i), (ii), (iii) and (iv) will come up in several places below. We call it an *extended* triangulation. Let

$$u_S = {c_{e,S}}^*(u) \in \mathcal{F}(\Delta_e(S)).$$

The total ordering of $\mathbb{T}$ leads to an identification of each $\Delta_e(S)$ with a standard extended simplex. Consequently each u_S becomes a simplex of the simplicial set $\underline{n} \mapsto \mathcal{F}(\Delta^n_e)$. We then have a unique map $\xi(u)\colon X \to |\mathcal{F}|$ such that, for each S as above with $|S| = n+1$, the diagram

$$\begin{tikzcd} \Delta(S) \arrow[r, "\cong"] \arrow[d, "c_S"] & \Delta^n \arrow[d, "\mathrm{char}(u_S)"] \\ X \arrow[r, "\xi(u)"] & |\mathcal{F}| \end{tikzcd}$$

commutes, where $\mathrm{char}(u_S)$ is the characteristic map associated to $u_S \in \mathcal{F}(\Delta^n_e)$. It is straightforward to show that the resulting homotopy class of maps from X to $|\mathcal{F}|$ depends only on the concordance class of $u \in \mathcal{F}(X)$.

We remark that $\xi\colon \mathcal{F}[X] \to [X, |\mathcal{F}|\,]$ so defined is a natural transformation. Indeed if $f\colon X \to Y$ is a smooth embedding, then $f^*\xi = \xi f^*$ by inspection. Any morphism $g\colon X \to Y$ in $\mathscr{X}$ can be factored as pf, where $f\colon X \to Y \times \mathbb{R}^k$ is a smooth embedding for some k and $p\colon Y \times \mathbb{R}^k \to Y$ is the projection. Let $s\colon Y \to Y \times \mathbb{R}^k$ be any smooth section of p. Then $s^*\xi = \xi s^*$, and consequently $p^*\xi = \xi p^*$ since p is inverse to s in the homotopy category of $\mathscr{X}$. Therefore $g^*\xi = f^*p^*\xi = f^*\xi p^* = \xi f^*p^* = \xi g^*$.

The construction of an inverse ϑ for ξ uses a simplicial approximation principle which we now recall. To introduce notation for that, we suppose first that L is a simplicial complex with a totally ordered vertex set $\mathbb{T}$. For $n \geq 0$ let L^s_n be the set of order-preserving maps $f\colon \{0, 1, \ldots, n\} \to \mathbb{T}$ such that $\mathrm{im}(f)$ is a simplex of L. Then $n \mapsto L^s_n$ is a simplicial set L^s and the realization $|L^s|$ is homeomorphic to L.

Next, let K be any simplicial complex and let Q be a simplicial set. The simplicial approximation principle states that, for any homotopy class of maps from K to $|Q|$, there exist a subdivision L of K, with a total ordering of its vertex set, and a simplicial map $L^s \to Q$ such that the induced map from $|L^s| \cong L \cong K$ to $|Q|$ is in the prescribed homotopy class.

Next we construct $\vartheta\colon [X, |\mathcal{F}|\,] \longrightarrow \mathcal{F}[X]$. Let $g\colon X \to |\mathcal{F}|$ be given. By the above approximation principle, we may assume that X comes with a smooth (extended) triangulation, with totally ordered vertex set $\mathbb{T}$, and that g is the realization of a simplicial map from X^s to the simplicial set $n \mapsto \mathcal{F}(\Delta^n_e)$. In particular, each distinguished subset $S \subset \mathbb{T}$ with $|S|-1 = n$ determines a nondegenerate n-simplex y_S of X^s and then an element $g(y_S) \in \mathcal{F}(\Delta^n_e)$. We now use the smooth homotopy (h_t) which comes with the extended triangulation. Then for each $n \geq 0$ and each distinguished subset $R \subset \mathbb{T}$ with $|R| - 1 = n$, the composition

$$c_{e,R}^{-1} h_1\colon V_R \to \Delta_e(R) \cong \Delta^n_e$$

is defined for a sufficiently small open V_R containing $c_R(\Delta(R))$ and contained in $h_1^{-1}(c_R(\Delta_R))$. The pullback of $g(y_R) \in \mathcal{F}(\Delta^n_e)$ under this defines $z_R \in \mathcal{F}(V_R)$. The elements z_R are compatible and so, by the sheaf property, determine a unique element $\vartheta(g)$ of $\mathcal{F}(X)$. Again, it is straightforward to verify that the concordance class of $\vartheta(g)$ depends only on the homotopy class of g.

Proposition A.1. *The maps ξ and ϑ are inverses of each other.*

Proof. Let $u \in \mathcal{F}(X)$. We have $\vartheta\xi(u) = h_1^*(u)$. Since h_1 is smoothly homotopic to $h_0 = \mathrm{id}_X$, this implies that $\vartheta\xi(u)$ is indeed concordant to u. Therefore

$$\vartheta\xi = \mathrm{id}\colon \mathcal{F}[X] \longrightarrow \mathcal{F}[X].$$

To show that $\xi\vartheta$ is the identity on $[X, |\mathcal{F}|\,]$, we can assume that $g\colon X \to |\mathcal{F}|$ is induced by a simplicial map from X^s to the simplicial set $\mathcal{F}(\Delta^\bullet_e)$ and that the homotopy (h_t) has $h_t = 0$ for t close to 0 and $h_t = h_1$ for t close to 1. Define $H\colon X \times \mathbb{R} \to X$ by

$$H(x,t) = \begin{cases} h_t(x) & t \in [0,1] \\ h_1(x) & t \geq 1 \\ h_0(x) & t \leq 0\,. \end{cases}$$

We introduce the notation $\mathcal{F}^{\mathbb{R}}$ for the sheaf $Y \mapsto \mathcal{F}(Y \times \mathbb{R})$ on $\mathscr{X}$, and note that the embeddings $y \mapsto (y,0)$ and $y \mapsto (y,1)$ of Y in $Y \times \mathbb{R}$ determine maps of sheaves $\mathrm{ev}_0\ , \mathrm{ev}_1\colon \mathcal{F}^{\mathbb{R}} \to \mathcal{F}$. We get a simplicial map G from X^s to $\mathcal{F}^{\mathbb{R}}(\Delta^\bullet_e)$, and consequently a map $|G|\colon X \to |\mathcal{F}^{\mathbb{R}}|$. Namely, for a nondegenerate n-simplex y_S of X^s let $G(y_S) \in \mathcal{F}^{\mathbb{R}}(\Delta^n_e)$ be the pullback of $g(y_S) \in \mathcal{F}(\Delta^n_e)$ along

$$(c_{e,S})^{-1} \circ H \circ (c_{e,S} \times \mathrm{id}_{\mathbb{R}})\colon \Delta^n_e \times \mathbb{R} \longrightarrow \Delta^n_e\,,$$

where we identify Δ^n_e with $\Delta_e(S)$ as usual. Lemma A.2 below implies that $g = |\mathrm{ev}_0 G|$ and $\xi\vartheta(g) = |\mathrm{ev}_1 G|$ are homotopic. $\square$

Lemma A.2. *The evaluation maps* $|ev_0|, |ev_1| : |\mathcal{F}^{\mathbb{R}}| \to |\mathcal{F}|$ *are homotopic.*

Proof. For an order-preserving map $f : \underline{n} \to \underline{1}$ let $\bar{f} : \Delta^n_e \to \Delta^n_e \times \mathbb{R}$ be the unique affine embedding which takes a vertex v of Δ^n to $(v, f(v))$. The formula $(u, f) \mapsto \bar{f}^*(u)$ determines a simplicial homotopy, i.e., a simplicial map from $n \mapsto \mathcal{F}(\Delta^n_e \times \mathbb{R}) \times \mathrm{mor}_\Delta(\underline{n}, \underline{1})$ to $n \mapsto \mathcal{F}(\Delta^n)$. The homotopy connects ev_0 with ev_1. □

Proof of Proposition 2.17. The special case where the closed subset A is empty is covered by Proposition A.1. The proof of the general case follows the same lines. To construct $\xi[u]$ for $u \in \mathcal{F}(X, A; z)$, we choose a smooth triangulation of X where each simplex which meets A is contained in a fixed open neighborhood Y of A with $u|Y = z$. Conversely, for a relative homotopy class of maps $X \to |\mathcal{F}|$ taking A to z, we can find a smooth triangulation of X with totally ordered vertex set and a simplicial map from X^s to $n \mapsto \mathcal{F}(\Delta^n_e)$ taking every nondegenerate simplex of X^s which meets A to z, and representing the relative homotopy class. □

A.2. *Categorical properties.*

Proposition A.3. *The construction* $\mathcal{F} \mapsto |\mathcal{F}|$ *takes pullback squares of sheaves to pullback squares of compactly generated Hausdorff spaces. In particular it respects products.*

Proof. The functor $\mathcal{F} \to |\mathcal{F}|$ is a composition of two functors: one from sheaves to simplicial sets, and another from simplicial sets to compactly generated Hausdorff spaces. It is obvious that the first of these respects pullbacks. The second also respects pullbacks by [9, §3, Thm. 3.1]. □

Definition A.4. The categorical coproduct $\mathcal{F}_1 \amalg \mathcal{F}_2$ of two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathscr{X}$ can be defined by $(\mathcal{F}_1 \amalg \mathcal{F}_2)(X) = \prod_i \mathcal{F}_1(X_i) \amalg \mathcal{F}_2(X_i)$ where X_i denotes the path component of X corresponding to an $i \in \pi_0(X)$.

Since Δ^n_e is path connected, we have

Proposition A.5. $|\mathcal{F}_1 \amalg \mathcal{F}_2| \cong |\mathcal{F}_1| \amalg |\mathcal{F}_2|$. □

Proposition A.6. *Suppose given sheaves* $\mathcal{E}, \mathcal{F}, \mathcal{G}$ *on* $\mathscr{X}$ *and morphisms* (*alias natural transformations*) $u : \mathcal{E} \to \mathcal{G}$, $v : \mathcal{F} \to \mathcal{G}$. *Let* $\mathcal{E} \times_{\mathcal{G}} \mathcal{F}$ *be the fiber product* (*pullback*) *of* u *and* v. *If* u *has the concordance lifting property, Definition* 4.5, *then the projection* $\mathcal{E} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{F}$ *has the concordance-lifting property and the following square is homotopy cartesian*:

$$\begin{tikzcd} |\mathcal{E} \times_{\mathcal{G}} \mathcal{F}| \arrow[r] \arrow[d] & |\mathcal{F}| \arrow[d, "v"] \\ |\mathcal{E}| \arrow[r, "u"] & |\mathcal{G}|. \end{tikzcd}$$

We begin with a special case of Proposition A.6, the case where $\mathcal{F} = \star$. Suppose that $u\colon \mathcal{E} \to \mathcal{G}$ has the concordance lifting property. Let z be a point in $\mathcal{G}(\star)$ and let $\mathcal{E}_z$ be the fiber of u over z (in the category of sheaves). Let $\mathrm{hofiber}_z\,|u|$ denote the homotopy fiber of $|u|\colon|\mathcal{E}| \to |\mathcal{G}|$ over the point z.

LEMMA A.7. *For any $y \in \mathcal{E}_z(\star)$, the homotopy set $\pi_n(\mathcal{E}_z, y)$ is in canonical bijection with $\pi_n(\mathrm{hofiber}_z\,|u|, y)$.*

Proof. The concordance lifting property gives that elements of $\pi_n(\mathcal{E}_z, y)$ are represented by pairs $(s,h) \in \mathcal{E}(S^n) \times \mathcal{G}(S^n \times \mathbb{R})$, where s has the value y near the base point of S^n and h is a concordance (relative to a neighborhood of the base point) from $u(s)$ to the constant z. It follows that $\pi_n(\mathcal{E}_z, y)$ is a relative homotopy group (set) of the map $|u|\colon|\mathcal{E}| \to |\mathcal{G}|$, which in turn is a homotopy group (set) of the homotopy fiber of $|u|$ over z. □

COROLLARY A.8. *In the situation of Lemma* A.7, *the sequence*

$$|\mathcal{E}_z| \hookrightarrow |\mathcal{E}| \xrightarrow{|u|} |\mathcal{G}|$$

is a homotopy fiber sequence.

Proof. The composite map from $|\mathcal{E}_z|$ to $|\mathcal{G}|$ is constant. This leads to a canonical map from $|\mathcal{E}_z|$ to the homotopy fiber of $|u|\colon|\mathcal{E}| \to |\mathcal{G}|$ over z. It is easy to verify directly that this induces a surjection on π_0. For each $y \in \mathcal{E}_z(\star)$, the induced map of homotopy sets

$$\pi_n(\mathcal{E}_z, y) \longrightarrow \pi_n(\mathrm{hofiber}_z\,|u|, y)$$

is the one from Lemma A.7. It is therefore always a bijection. □

Proof of Proposition A.6. We fix $z \in \mathcal{F}(\star)$ and obtain $v(z) \in \mathcal{G}(\star)$. The fiber of

$$\mathcal{E} \times_{\mathcal{G}} \mathcal{F} \longrightarrow \mathcal{F}$$

over z is identified with the fiber of $u\colon\mathcal{E} \to \mathcal{G}$ over $v(z)$. Using Corollary A.8 we can conclude that the homotopy fiber of $|\mathcal{E} \times_{\mathcal{G}} \mathcal{F}| \longrightarrow |\mathcal{F}|$ over z maps to the homotopy fiber of $|u|\colon|\mathcal{E}| \to |\mathcal{G}|$ over $v(z)$ by a homotopy equivalence. □

A.3. *Cocycle sheaves and classifying spaces.* This section contains the proof of Theorem 4.2. To prepare for this we start with a variation on the standard nerve construction. Recall that $\mathscr{D}\underline{n}$ is the poset of nonempty subsets of $\underline{n} = \{0,1,2,\dots,n\}$. There are functors $v_n\colon \mathscr{D}\underline{n} \to \underline{n}$ given by $v_n(S) = \max(S) \in \underline{n}$.

LEMMA A.9. *Let $\mathscr{C}$ be a small category. Then the map of simplicial sets*

$$(\,n \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C})) \quad\longrightarrow\quad (\,n \mapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathscr{C}))$$

given by composition with $v_\bullet$ induces a homotopy equivalence of the geometric realizations.

Proof. The simplicial set $(n \mapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathscr{C}))$ is obtained by applying Kan's functor *ex*, which is right adjoint to the barycentric subdivision, to $(n \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C}))$. The statement is therefore a special case of [21, 3.7]. $\square$

We note that the simplicial set $(n \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C}))$ is precisely the nerve of $\mathscr{C}$, denoted $N_\bullet\mathscr{C}$ in Section 4.

Let J be any infinite set and $\mathrm{emb}(\underline{n}, J)$ the set of injective maps. This defines a Δ-set (or incomplete simplicial set) $n \mapsto \mathrm{emb}(\underline{n}, J)$; cf. [38].

LEMMA A.10. *Let J be an infinite set and let $K_\bullet$ be a simplicial set. The geometric realization $|K_\bullet|$ is homotopy equivalent to the geometric realization of the Δ-set $n \mapsto K_n \times \mathrm{emb}(\underline{n}, J)$.*

Proof. There is a projection p from the realization of $n \mapsto K_n \times \mathrm{emb}(\underline{n}, J)$ as a Δ-set to the realization $|K_\bullet|$ of $K_\bullet$ as a simplicial set. We will show that p has contractible fibers. Let y be a point in the m-skeleton $|K_\bullet|^m$ of $|K_\bullet|$, but not in the $(m-1)$-skeleton. Then $p^{-1}(y)$ is homeomorphic to the classifying space of the poset $\mathscr{R}$ whose elements are the nonempty finite subsets of J equipped with a total ordering and an order-preserving surjection to $\underline{m}$. For each finite subset $\mathscr{R}'$ of $\mathscr{R}$, there exists $T \in \mathscr{R}$ which is disjoint from all $T' \in \mathscr{R}'$, so that $T' \leq T' \cup T \geq T$ in $\mathscr{R}$ where $T' \cup T$ has the concatenated ordering. Hence $B\mathscr{R}'$ is contractible in $B\mathscr{R}$, and so $B\mathscr{R} \cong p^{-1}(y)$ is contractible.

It is easy to refine this argument to an induction proof showing that p restricts to a homotopy equivalence $p^{-1}(|K_\bullet|^m) \to |K_\bullet|^m$ for $m = 0, 1, 2, \ldots$. We omit the details. $\square$

In the following we use double vertical bars $\|...\|$ for the geometric realization of Δ-sets.

COROLLARY A.11. *Let J be the fixed infinite set from Definition* 4.1. *The space $B|\mathcal{F}|$ is homotopy equivalent to the geometric realization $\|\hat{\mathcal{F}}_\bullet\|$, where $\hat{\mathcal{F}}_\bullet$ is the Δ-set defined by $\hat{\mathcal{F}}_n = \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^n_e)) \times \mathrm{emb}(\underline{n}, J)$.*

Proof. We consider the map of bisimplicial sets

$$\hom(\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^m_e) \longrightarrow \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^m_e)).$$

The geometric realization in the n-direction is a homotopy equivalence for each m by Lemma A.9, and the map of geometric realizations of the bisimplicial sets is then also a homotopy equivalence. But then the geometric realization of the map between the corresponding diagonal simplicial sets is a homotopy equivalence. $\square$

We turn to the construction of a comparison map Ψ from $\hat{\mathcal{F}}_\bullet$ in Corollary A.11 to the simplicial set $\underline{n} \mapsto \beta\mathcal{F}(\Delta^n_e)$. An n-simplex in $\hat{\mathcal{F}}_\bullet$ is a pair

$$\varphi \colon \mathscr{D}\underline{n}^{\mathrm{op}} \longrightarrow \mathcal{F}(\Delta^n_e)\ , \qquad \lambda \in \mathrm{emb}(\underline{n}, J).$$

The pair (φ, λ) carries exactly the same information as an element in $\beta\mathcal{F}(\Delta^n_e)$ whose underlying J-indexed open covering is given by $j \mapsto \Delta^n_e$ if $j = \lambda(t)$ for some $t \in \underline{n}$ and $j \mapsto \emptyset$ otherwise. To make these data compatible with face operators, we need to replace the nonempty open sets in the open covering by smaller ones, according to the rule

$$j = \lambda(t) \quad \mapsto \quad \{ (x_0, x_1, \ldots, x_n) \in \Delta^n_e \mid x_t > 0\}. \tag{A.1}$$

The remaining data can be restricted and we now have an element $\Psi(\varphi, \lambda)$ in $\beta\mathcal{F}(\Delta^n_e)$, and hence a map

$$\Psi : B|\mathcal{F}| \simeq \|\hat{\mathcal{F}}_\bullet\| \longrightarrow |\beta\mathcal{F}|. \tag{A.2}$$

We proceed to the construction of a natural map

$$\Lambda : \beta\mathcal{F}[X] \longrightarrow [X, \|\hat{\mathcal{F}}_\bullet\|\,] \tag{A.3}$$

which will define a homotopy inverse to Ψ.

Let $(\mathscr{Y}, \varphi_{\bullet\bullet})$ be an element of $\beta\mathcal{F}(X)$ with $\mathscr{Y} = (Y_j)_{j\in J}$. We choose a smooth triangulation of X with the extra structure from Section A.1, with totally ordered vertex set $\mathbb{T}$. We assume $\mathbb{T} \subset J$. For each $v \in \mathbb{T}$ let

$$\mathrm{star}_e(v) = \bigcup_{S \ni v} c_{e,S}(\Delta_e(S))$$

where S runs through the simplices having v as a vertex. We assume that the covering of X by these open sets is subordinate to the covering $\mathscr{Y}$ in the sense that

$$\mathrm{star}_e(v) \subset Y_{\kappa(v)}\ , \qquad v \in \mathbb{T},$$

for some map $\kappa : J \to J$. Then for distinguished subsets Q, R, S of $\mathbb{T}$ with $Q \subset R \subset S$, the pullback under $c_{e,S}$ of the morphism $\varphi_{\kappa(Q)\kappa(R)}$ in $\mathcal{F}(Y_{\kappa(R)})$ is a morphism in $\mathcal{F}(\Delta_e(S))$. Together these morphisms define an element

$$x_S \in \hom(\mathscr{D}(S)^{\mathrm{op}}, \mathcal{F}(\Delta_e(S)))\,.$$

With the embedding $S \subset \mathbb{T} \to J$, the element x_S becomes an n-simplex (where $n+1 = |S|$) of $\hat{\mathcal{F}}_\bullet$. As these simplices x_S are compatibly constructed they determine a map from X to $\|\hat{\mathcal{F}}_\bullet\|$. It follows from Lemma A.10 that the homotopy class of that map does not depend on the way in which the vertex set of the triangulation is embedded in J, and then it is altogether clear that the homotopy class depends only on the concordance class of $(\mathscr{Y}, \varphi_{\bullet\bullet}) \in \beta\mathcal{F}(X)$. Hence we have defined $\Lambda : \beta\mathcal{F}[X] \to [X, \|\hat{\mathcal{F}}_\bullet\|\,]$.

THEOREM A.12. *The maps Ψ and Λ of* (A.2) *and* (A.3) *define reciprocal homotopy equivalences between $B|\mathcal{F}|$ and $|\beta\mathcal{F}|$.*

Proof. Suppose that an element of $\beta\mathcal{F}[X]$ is represented by a pair $(\mathscr{Y},\varphi_{\bullet\bullet})$, where $\mathscr{Y}$ is a J-indexed open covering of X. Then, by construction and inspection, $\Psi\Lambda$ of that element is represented by a pair $(\mathscr{Y}',\varphi'_{\bullet\bullet})$ for which $\kappa\colon J\to J$ can be found such that $Y'_j\subset Y_{\kappa(j)}$ for $j\in J$ and φ'_{RS} is the restriction of $\varphi_{\kappa(R)\kappa(S)}$ to Y'_S, for finite nonempty $R,S\subset J$ with $R\subset S$. (What makes the inspection slightly difficult is that the identification $\vartheta\colon[X,|\beta\mathcal{F}|\,]\to\beta\mathcal{F}[X]$ of Section A.1 is also involved.) Thus we have a situation where one element of $\beta\mathcal{F}(X)$ "refines" another. Lemma A.13 below then guarantees that the two elements are concordant. Hence $\Psi\Lambda=\mathrm{id}$ on homotopy sets $\beta\mathcal{F}[X]$.

Next we show that $\Lambda\colon\beta\mathcal{F}[X]\to[X,\|\hat{\mathcal{F}}_\bullet\|\,]$ is onto for any X in $\mathscr{X}$. Any element of $[X,\|\hat{\mathcal{F}}_\bullet\|\,]$ can be represented by a simplicial map $f\colon X^s\to\hat{\mathcal{F}}_\bullet$ where X^s is the simplicial set associated to some smooth triangulation of X with totally ordered vertex set $\mathbb{T}$. We assume that $\mathbb{T}\subset J$ and that the triangulation comes with the "extended" data of A.1. For $j\in\mathbb{T}\subset J$ let Y_j be a sufficiently small open neighborhood of the union of all simplices having j as a vertex. For all other $j\in J$ let $Y_j=\emptyset$. For distinguished $Q,R,S\subset\mathbb{T}$ with $Q\subset R\subset S$ the data in f provide a morphism in $\mathcal{F}(\Delta_e(S))$, corresponding to the inclusion $Q\subset R$ (of nonempty subsets of the totally ordered S). Pull this back along

$$c_S^{-1}h_1\colon V_S\to\Delta(S)\subset\Delta_e(S)$$

where $V_S\subset X$ is some open neighborhood of the simplex $c_S(\Delta(S))$ such that $h_1(V_S)$ is contained in the simplex. The result is a morphism in $\mathcal{F}(V_S)$. Keeping Q and R fixed, note the compatibility of these morphisms as S runs through the distinguished subsets of $\mathbb{T}$ containing R. By the sheaf property this leads to a single morphism in

$$\mathcal{F}(\bigcup_{S\supset R}V_S)$$

which we can restrict to obtain $\varphi_{QR}\in\mathcal{F}(Y_R)$. Indeed if the Y_j are small enough, then Y_R will be contained in the union of the V_S for $S\supset R$. We have therefore constructed an open covering $\mathscr{Y}=(Y_j)_{j\in J}$ of X and elements $\varphi_{QR}\in\mathcal{F}(Y_R)$ such that $(\mathscr{Y},\varphi_{\bullet\bullet})\in\beta\mathcal{F}(X)$. Following the instructions above for finding a representative for Λ of $(\mathscr{Y},\varphi_{\bullet\bullet})$, we get a map which is homotopic to f by the argument which we saw in the second part of the proof of Proposition A.1. The conclusion is that Λ is indeed surjective.

The final step is to note that Ψ, as a map from $|\hat{\mathcal{F}}_\bullet|$ to $|\beta\mathcal{F}|$, induces a surjection $\pi_1(|\hat{\mathcal{F}}_\bullet|,z)\to\pi_1(|\beta\mathcal{F}|,\Psi(z))$ for any choice of base vertex $z\in|\hat{\mathcal{F}}_\bullet|$. We leave this verification to the reader: Given an element u in $\pi_1(|\beta\mathcal{F}|,\Psi(z))$, an element v of $\pi_1(|\hat{\mathcal{F}}_\bullet|,z)$ can be obtained by applying the procedure Λ above to u in a relative form. The relative case of Lemma A.13 below implies $\Psi(v)=u$.

It is a formality to show that a map $q\colon C \to D$ between CW-spaces which induces bijections $[X,C] \to [X,D]$ for every X in $\mathscr{X}$ and surjections

$$\pi_1(C,z) \to \pi_1(D,q(z))$$

for every $z \in C$ induces bijections $\pi_n(C,z) \to \pi_n(D,q(z))$ for $n \geq 0$ and $z \in C$. Such a map is therefore a homotopy equivalence. We have just verified that this criterion applies with $q = \Psi$, showing that $\Psi\colon \|\hat{\mathcal{F}}_\bullet\| \to |\beta\mathcal{F}|$ is a homotopy equivalence. $\square$

LEMMA A.13. *Let $(\mathscr{Y},\varphi_{\bullet\bullet})$ and $(\mathscr{Y}',\varphi'_{\bullet\bullet})$ be elements of $\beta\mathcal{F}(X)$. Suppose that there exists a map $\kappa\colon J \to J$ such that $Y'_j \subset Y_{\kappa(j)}$ for all $j \in J$, and φ'_{RS} is the restriction of $\varphi_{\kappa(R)\kappa(S)}$ to Y'_S, for all finite nonempty $R,S \subset J$ with $R \subset S$. Then $(\mathscr{Y},\varphi_{\bullet\bullet})$ and $(\mathscr{Y}',\varphi'_{\bullet\bullet})$ are concordant. If $(\mathscr{Y},\varphi_{\bullet\bullet})$ and $(\mathscr{Y}',\varphi'_{\bullet\bullet})$ are in $\beta\mathcal{F}(X,A;z)$ for some closed $A \subset X$ and some $z \in \beta\mathcal{F}(\star)$, and if $\kappa(j) = j$ for all $j \in J$ such that the closure of Y_j has nonempty intersection with A, then the concordance can be taken relative to A.*

Proof. We assume first that the fixed indexing set J is uncountable, rather than just infinite, and concentrate on the absolute case, $A = \emptyset$.

The case where $\kappa = \mathrm{id}_J$ is straightforward. Hence $(\mathscr{Y}',\varphi'_{\bullet\bullet})$ is concordant to $(\mathscr{Y}'',\varphi''_{\bullet\bullet})$ where $Y''_j = Y_{\kappa(j)}$ and $\varphi''_{RS} = \varphi_{\kappa(R)\kappa(S)}$. It remains to find a concordance from $(\mathscr{Y}'',\varphi''_{\bullet\bullet})$ to $(\mathscr{Y},\varphi_{\bullet\bullet})$. Alternatively, to keep notation under control, we may assume from now on that $(\mathscr{Y}',\varphi'_{\bullet\bullet}) = (\mathscr{Y}'',\varphi''_{\bullet\bullet})$; in other words $Y'_j = Y_{\kappa(j)}$ for all $j \in J$.

The sets $\{j \in J \mid Y'_j \neq \emptyset\}$ and $\{i \in J \mid Y_i \neq \emptyset\}$ are countable, since the coverings $\mathscr{Y}'$ and $\mathscr{Y}$ are locally finite and X admits a countable base. Hence there exists a bijection $\lambda\colon J \to J$ such that $Y_{\lambda(j)} \cap Y_j = \emptyset = Y_{\lambda(j)} \cap Y_{\kappa(j)}$ for all $j \in J$; for example, λ can be chosen so that $Y_\lambda(j) = \emptyset$ if $Y_j \neq \emptyset$ or $Y_{\kappa(j)} \neq \emptyset$. Now let

$$W_j = \big(Y_j \times\,]-\infty, 1/2[\big) \cup \big(Y_{\lambda(j)} \times\,]1/4, 3/4[\big) \cup \big(Y_{\kappa(j)} \times\,]1/2, \infty[\big).$$

The W_j for $j \in J$ constitute an open covering $\mathscr{W}$ of $X \times \mathbb{R}$. For any finite nonempty $S \subset J$, we have a decomposition of W_S into disjoint open sets

$$\begin{array}{lll} Y_S \times\,]-\infty, 1/2[\,, & Y_{\lambda(S)} \times\,]1/4, 3/4[\,, & Y_{\kappa(S)} \times\,]1/2, \infty[\,, \\ Y_{Q \cup \lambda(S \smallsetminus Q)} \times\,]1/4, 1/2[\,, & Y_{\lambda(Q) \cup \kappa(S \smallsetminus Q)} \times\,]1/2, 3/4[\,, & \end{array}$$

where Q runs through the nonempty proper subsets of S. Therefore, given finite nonempty $R,S \subset J$ with $R \subset S$, there is a unique morphism ψ_{RS} in $\mathcal{F}(W_S)$ whose restrictions to the various summands of W_S in the above decomposition are the pullbacks of φ_{RS}, $\varphi_{\lambda(R)\lambda(S)}$, $\varphi_{\kappa(R)\kappa(S)}$, etc. etc., under the projections to Y_S, $Y_{\lambda(S)}$, $Y_{\kappa(S)}$, $Y_{Q \cup \lambda(S \smallsetminus Q)}$ and $Y_{\lambda(Q) \cup \kappa(S \smallsetminus Q)}$, respectively. (Here the two "etc." are short for φ_{TU} where $U = Q \cup \lambda(S \smallsetminus Q)$ and $T = (R \cap Q) \cup \lambda(R \smallsetminus Q)$ in the first case, while $U = \lambda(Q) \cup \kappa(S \smallsetminus Q)$ and $T =$

$\lambda(R \cap Q) \cup \kappa(R \smallsetminus Q)$ in the second case.) Clearly $(\mathscr{W}, \psi_{\bullet\bullet})$ is a concordance from $(\mathscr{Y}, \varphi_{\bullet\bullet})$ to $(\mathscr{Y}', \varphi'_{\bullet\bullet})$.

Next we look at the relative case, $A \neq \emptyset$, but continue to assume that J is uncountable. As in the absolute case we may assume that $Y'_j = Y_{\kappa(j)}$ for all $j \in J$. Choose a bijection $\lambda\colon J \to J$ such that $\lambda(j) = j$ whenever $\kappa(j) = j$, and such that $Y_j \cap Y_{\lambda(j)} = \emptyset = Y_{\kappa(j)} \cap Y_{\lambda(j)}$ for the remaining j. Again let

$$W_j = \big(Y_j \times\,]-\infty, 1/2[\,\big) \cup \big(Y_{\lambda(j)} \times\,]1/4, 3/4[\,\big) \cup \big(Y_{\kappa(j)} \times\,]1/2, \infty[\,\big).$$

The W_j for $j \in J$ constitute an open covering $\mathscr{W}$ of $X \times \mathbb{R}$. For a finite nonempty $S \subset J$ which is contained in the fixed point set of κ, we simply have $W_S = Y_S \times \mathbb{R}$. For a finite nonempty $S \subset J$ which does not contain any fixed points of κ, we have a decomposition of W_S into disjoint open sets

$$\begin{array}{lll} Y_S \times\,]-\infty, 1/2[\,, & Y_{\lambda(S)} \times\,]1/4, 3/4[\,, & Y_{\kappa(S)} \times\,]1/2, \infty[\,, \\ Y_{Q \cup \lambda(S \smallsetminus Q)} \times\,]1/4, 1/2[\,, & Y_{\lambda(Q) \cup \kappa(S \smallsetminus Q)} \times\,]1/2, 3/4[\,, & \end{array}$$

as before, where Q runs through the nonempty proper subsets of S. For finite nonempty $S \subset J$ which contains some fixed points of κ and some nonfixed points of κ, write $S = S_1 \cup S_2$ where $S_1 = \{j \in S \mid \kappa(j) = j\}$ and $S_2 = S \smallsetminus S_1$. Then W_{S_2} decomposes into disjoint open sets as above, whereas $W_{S_1} = Y_{S_1} \times \mathbb{R}$. Hence $W_S = W_{S_1} \cap W_{S_2}$ still decomposes as a disjoint union of open sets

$$\begin{array}{lll} Y_S \times\,]-\infty, 1/2[\,, & Y_{\lambda(S)} \times\,]1/4, 3/4[\,, & Y_{\kappa(S)} \times\,]1/2, \infty[\,, \\ Y_{Q \cup \lambda(S \smallsetminus Q)} \times\,]1/4, 1/2[\,, & Y_{\lambda(Q) \cup \kappa(S \smallsetminus Q)} \times\,]1/2, 3/4[\,, & \end{array}$$

where Q runs through the nonempty proper subsets of S_2 only. We can therefore define morphisms ψ_{RS} in $\mathcal{F}(W_S)$ much as in the absolute case and obtain a relative concordance $(\mathscr{W}, \psi_{\bullet\bullet})$ from $(\mathscr{Y}, \varphi_{\bullet\bullet})$ to $(\mathscr{Y}', \varphi'_{\bullet\bullet})$.

Now we must consider the case(s) where J is countably infinite. We can reason as before provided that X is a closed manifold, because in that case the sets $\{j \in J \mid Y'_j \neq \emptyset\}$ and $\{i \in J \mid Y_j \neq \emptyset\}$ are finite. While this is not exactly what we want, it allows us to make a comparison between the case where J is countable and the case where it is uncountable. Choose an uncountable set J^u containing J as a subset. Corresponding to J and J^u we have two variants of $\beta\mathcal{F}$. We keep the notation $\beta\mathcal{F}$ for the J-variant, and write $\beta^u\mathcal{F}$ for the J^u-variant. There is a natural inclusion $\beta\mathcal{F}(X) \to \beta^u\mathcal{F}(X)$, because any J-indexed open covering of X can be regarded as a J^u-indexed covering of X where all open sets with labels in $J^u \smallsetminus J$ are empty. By all the above, $|\beta\mathcal{F}| \to |\beta^u\mathcal{F}|$ induces an isomorphism of homotopy groups or homotopy sets, for any choice of base vertex in $|\beta\mathcal{F}|$, the point being that spheres are closed manifolds. By Proposition 2.17, this implies that the inclusion-induced map of concordance sets

$$\beta\mathcal{F}[X, A; z] \longrightarrow \beta^u\mathcal{F}[X, A; z]$$

is always a bijection, and not just when X is closed. We have therefore reduced the case of a countable J to the case of an uncountable one. $\square$

B. Realization and homotopy colimits

B.1. *Realization and squares.*

LEMMA B.1. *Let $u_\bullet : E_\bullet \longrightarrow B_\bullet$ be a map between incomplete simplicial spaces* (*or good simplicial spaces*)*. Suppose that the squares*

$$\begin{array}{ccc} E_k & \xrightarrow{u_k} & B_k \\ \big\downarrow{\scriptstyle d_i} & & \big\downarrow{\scriptstyle d_i} \\ E_{k-1} & \xrightarrow{u_{k-1}} & B_{k-1} \end{array}$$

are all homotopy cartesian ($k \geq i \geq 0$)*. Then the following is also homotopy cartesian*:

$$\begin{array}{ccc} E_0 & \xrightarrow{u_0} & B_0 \\ \big\downarrow{\scriptstyle \text{incl.}} & & \big\downarrow{\scriptstyle \text{incl.}} \\ |E_\bullet| & \xrightarrow{|u_\bullet|} & |B_\bullet|. \end{array}$$

LEMMA B.2. *Let $u_\bullet : E_\bullet \longrightarrow B_\bullet$ be a map between incomplete simplicial spaces* (*or good simplicial spaces*)*. Suppose that, in each square*

$$\begin{array}{ccc} E_k & \xrightarrow{u_k} & B_k \\ \big\downarrow{\scriptstyle d_i} & & \big\downarrow{\scriptstyle d_i} \\ E_{k-1} & \xrightarrow{u_{k-1}} & B_{k-1} \end{array}$$

the canonical map from any homotopy fiber of u_k to the corresponding homotopy fiber of u_{k-1} induces an isomorphism in integer homology. Then in the square

$$\begin{array}{ccc} E_0 & \xrightarrow{u_0} & B_0 \\ \big\downarrow{\scriptstyle \text{incl.}} & & \big\downarrow{\scriptstyle \text{incl.}} \\ |E_\bullet| & \xrightarrow{|u_\bullet|} & |B_\bullet|, \end{array}$$

the canonical map from any homotopy fiber of u_0 to the corresponding homotopy fiber of $|u_\bullet|$ induces an isomorphism in integer homology.

Proofs. It is shown in [39, 1.6] and [25, Prop.4] that the geometric realization procedure for simplicial spaces respects degree-wise quasifibrations and homology fibrations under reasonable conditions. The two lemmas follow from these statements upon converting the maps u_k into fibrations. □

COROLLARY B.3. *Let $\mathscr{C}$ be a small category and let $u : \mathcal{G}_1 \to \mathcal{G}_2$ be a natural transformation between functors from $\mathscr{C}$ to spaces. Suppose that, for each morphism $f : a \to b$ in $\mathscr{C}$, the map f_* from any homotopy fiber of u_a

to the corresponding homotopy fiber of u_b *induces an isomorphism in integer homology. Then for each object* a *of* $\mathscr{C}$, *the inclusion of any homotopy fiber of* u_a *in the corresponding homotopy fiber of* $u_*\colon \operatorname{hocolim}\mathcal{G}_1 \to \operatorname{hocolim}\mathcal{G}_2$ *induces an isomorphism in integer homology.*

Proof. Apply Lemma B.2 with $E_k := \coprod \mathcal{G}_1(D(k))$ and $B_k = \coprod \mathcal{G}_2(D(k))$, where both coproducts run over the set of contravariant functors D from the poset $\underline{k}$ to $\mathscr{C}$. Then $|E_\bullet|$ is $\operatorname{hocolim}\mathcal{G}_1$ and $|B_\bullet|$ is $\operatorname{hocolim}\mathcal{G}_2$. □

B.2. *Homotopy colimits.* Any functor $\mathcal{D}$ from a small (discrete) category $\mathscr{C}$ to the category of spaces has a *colimit*, $\operatorname{colim}\mathcal{D}$. This is the quotient space of the coproduct

$$\coprod_{a \text{ in } \mathscr{C}} \mathcal{D}(a)$$

obtained by identifying $x \in \mathcal{D}(a)$ with $f_*(x) \in \mathcal{D}(b)$ for any morphisms $f\colon a \to b$ in $\mathscr{C}$ and elements $x \in \mathcal{D}(a)$. It is well known that the colimit construction is not well behaved from a homotopy theoretic point of view. Namely, suppose that $w\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a natural transformation between functors from $\mathscr{C}$ to spaces and that $w_a : \mathcal{D}_1(a) \to \mathcal{D}_2(a)$ is a homotopy equivalence for any object a in $\mathscr{C}$. Then this does not in general imply that the map induced by w from $\operatorname{colim}\mathcal{D}_1$ to $\operatorname{colim}\mathcal{D}_2$ is again a homotopy equivalence. (It is easy to make examples with $\mathscr{C}$ equal to the poset of proper subsets of a two-element set, so that the colimits become pushouts.)

Call a functor $\mathcal{D}$ from $\mathscr{C}$ to spaces *cofibrant* if, for any diagram of functors (from $\mathscr{C}$ to spaces) and natural transformations

$$\mathcal{D} \xrightarrow{\ v\ } \mathcal{E} \xleftarrow{\ w\ } \mathcal{F}$$

where $w_a\colon \mathcal{F}(a) \to \mathcal{E}(a)$ is a homotopy equivalence for all $a \in \mathscr{C}$, there exists a natural transformation $v'\colon \mathcal{D} \to \mathcal{F}$ and a *natural* homotopy $\mathcal{D}(a)\times[0,1] \to \mathcal{E}(a)$ (for all a) connecting wv' and v. It is not hard to show the following. If $v\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a natural transformation between cofibrant functors such that $v_a\colon \mathcal{D}_1(a) \to \mathcal{D}_2(a)$ is a homotopy equivalence for each $a \in \mathscr{C}$, then v has a natural homotopy inverse (with *natural* homotopies) and therefore the induced map $\operatorname{colim}\mathcal{D}_1 \to \operatorname{colim}\mathcal{D}_2$ is a homotopy equivalence.

This suggests the following procedure for making colimits homotopy invariant. Suppose that $\mathcal{D}$ from $\mathscr{C}$ to spaces is any functor. Try to find a natural transformation $\mathcal{D}' \to \mathcal{D}$ specializing to homotopy equivalences $\mathcal{D}'(a) \to \mathcal{D}(a)$ for all a in $\mathscr{C}$, where $\mathcal{D}'$ is cofibrant. Then define the *homotopy colimit* of $\mathcal{D}$ to be $\operatorname{colim}\mathcal{D}'$. If it can be done, $\operatorname{hocolim}\mathcal{D}$ is at least well defined up to homotopy equivalence.

This point of view is carefully presented in [5]. Some of the ideas go back to [27]. As we will see in a moment, there is a construction for $\mathcal{D}'$ which depends naturally on $\mathcal{D}$.

The standard foundational reference for homotopy colimits and homotopy limits is the book [2] by Bousfield and Kan. But the first explicit construction of homotopy colimits in general appears to be due to Segal [41].

Again let $\mathcal{D}$ be a functor from a discrete small category $\mathscr{C}$ to the category of spaces. Following Segal we introduce a topological category denoted $\mathscr{C}\!\int\!\mathcal{D}$, the *transport category* of $\mathcal{D}$:

$$\mathrm{ob}(\mathscr{C}\!\int\!\mathcal{D}) = \coprod_{a\in\mathrm{ob}(\mathscr{C})} \mathcal{D}(a)\,, \qquad \mathrm{mor}(\mathscr{C}\!\int\!\mathcal{D}) = \coprod_{f\in\mathrm{mor}(\mathscr{C})} \mathcal{D}(\sigma(f))\,.$$

Here $\sigma(f)$ denotes the source of a morphism f in $\mathscr{C}$. We will write morphisms in $\mathscr{C}\!\int\!\mathcal{D}$ as pairs (f,x) where $f\in\mathrm{mor}(\mathscr{C})$ and $x\in\mathcal{D}(\sigma(f))$. The composition $(g,y)\circ(f,x)$ of two such morphisms is defined if and only if $g\circ f$ is defined in $\mathscr{C}$ and $f_*(x)=y$, in which case $(g,y)\circ(f,x)=(g\circ f,x)$. The classifying space $B(\mathscr{C}\!\int\!\mathcal{D})$ is a model for the homotopy colimit of $\mathcal{D}$.

To relate $B(\mathscr{C}\!\int\!\mathcal{D})$ to our earlier discussion we define a functor $\mathcal{D}'$ from $\mathscr{C}$ to spaces as follows. For $a\in\mathrm{ob}(\mathscr{C})$ let $\mathscr{C}\!\downarrow\! a$ be the category of $\mathscr{C}$-objects over a, [23, II.6]. Let

$$\mathcal{D}'(a) := B\left((\mathscr{C}\!\downarrow\! a)\!\int\!\mathcal{D}\right)$$

for objects a in $\mathscr{C}$, where we view $\mathcal{D}$ as a functor on $\mathscr{C}\!\downarrow\! a$. Then $\mathcal{D}'$ is cofibrant and the canonical map $\mathcal{D}'(a)\to\mathcal{D}(a)$ is a homotopy equivalence for every a in $\mathscr{C}$. Moreover,

$$B(\mathscr{C}\!\int\!\mathcal{D}) \cong \mathrm{colim}\,\mathcal{D}'.$$

Note in passing that if $\mathcal{D}(a)$ is a singleton for each a in $\mathcal{C}$, then the transport category $\mathscr{C}\!\int\!\mathcal{D}$ is identified with $\mathscr{C}$ and so $\mathrm{hocolim}\,\mathcal{D}=B\mathscr{C}$.

Proposition B.4. *Let $w\colon\mathcal{D}_1\to\mathcal{D}_2$ be a natural transformation between functors from $\mathscr{C}$ to spaces. Suppose that $w_a:\mathcal{D}_1(a)\to\mathcal{D}_2(a)$ is a homotopy equivalence for any object a in $\mathscr{C}$. Then the map* $\mathrm{hocolim}\,\mathcal{D}_1\longrightarrow\mathrm{hocolim}\,\mathcal{D}_2$ *induced by w is a homotopy equivalence, where* $\mathrm{hocolim}\,\mathcal{D}_i=B(\mathscr{C}\!\downarrow\!\mathcal{D}_i)$.

This is just a partial summary of our conclusions above. We proceed to a reformulation, B.6 below, in which homotopy colimits are not mentioned explicitly.

Definition B.5. Let $p\colon\mathscr{E}\to\mathscr{C}$ be a continuous functor between small topological categories, where $\mathrm{ob}(\mathscr{C})$ and $\mathrm{mor}(\mathscr{C})$ are discrete. We say that p is a *transport projection* if the following is a pullback square of spaces:

$$\begin{tikzcd}
\mathrm{mor}(\mathscr{E}) \arrow[r, "\text{source}"] \arrow[d, "p"] & \mathrm{ob}(\mathscr{E}) \arrow[d, "p"] \\
\mathrm{mor}(\mathscr{C}) \arrow[r, "\text{source}"] & \mathrm{ob}(\mathscr{C})
\end{tikzcd}$$

Proposition B.6. *Let* $p\colon \mathscr{E} \to \mathscr{C}$ *and* $p'\colon \mathscr{E}' \to \mathscr{C}$ *be transport projections as in Definition* B.5. *Let* $u\colon \mathscr{E} \to \mathscr{E}'$ *be a continuous functor over* $\mathscr{C}$. *Suppose also that, for each object* c *in* $\mathscr{C}$, *the restriction* $\mathscr{E}_c \to \mathscr{E}'_c$ *of* u *to the fibers over* c *is a homotopy equivalence. Then* $Bu\colon B\mathscr{E} \to B\mathscr{E}'$ *is a homotopy equivalence.*

Proof. Note that $\mathscr{E} \cong \mathscr{C} \downarrow \mathcal{D}$ and $\mathscr{E}' \cong \mathscr{C} \downarrow \mathcal{D}$ where $\mathcal{D}(c) = \mathscr{E}_c$ and $\mathcal{D}'(c) = \mathscr{E}'_c$ for an object c in $\mathscr{C}$. Note also that $\mathscr{E}_c$ and $\mathscr{E}'_c$ are topological categories in which every morphism is an identity, that is, they are just spaces. □

Next we mention two useful naturality properties of homotopy colimits. To make a homotopy colimit, we need a pair $(\mathscr{C}, \mathcal{D})$ consisting of a small category $\mathscr{C}$ and a functor $\mathcal{D}$ from $\mathscr{C}$ to spaces. By a *morphism* from one such pair $(\mathscr{C}^s, \mathcal{D}^s)$ to another, $(\mathscr{C}^t, \mathcal{D}^t)$, we understand a pair $(\mathcal{F}, \nu)$ consisting of a functor $\mathcal{F}\colon \mathscr{C}^s \to \mathscr{C}^t$ and a natural transformation ν from $\mathcal{D}^s$ to $\mathcal{D}^t\mathcal{F}$.

Remark B.7. Such a morphism induces a map $(\mathcal{F}, \nu)_*$ from $\operatorname{hocolim} \mathcal{D}^s$ to $\operatorname{hocolim} \mathcal{D}^t$.

Let $(\mathcal{F}_0, \nu_0)$ and $(\mathcal{F}_1, \nu_1)$ be morphisms from $(\mathscr{C}^s, \mathcal{D}^s)$ to $(\mathscr{C}^t, \mathcal{D}^t)$. Let θ be a natural transformation from $\mathcal{F}_0$ to $\mathcal{F}_1$ such that $\nu_1 = \mathcal{D}^t(\theta) \circ \nu_0$.

Remark B.8. Such a θ induces a homotopy θ_* from $(\mathcal{F}_0, \nu_0)_*$ to $(\mathcal{F}_1, \nu_1)_*$.

Proof. Let $\mathscr{I} = \{0, 1\}$, be viewed as an ordered set with the usual order and then as a category. Then $B\mathscr{I} \cong [0, 1]$. Let $p\colon \mathscr{C} \times \mathscr{I} \to \mathscr{C}$ be the projection. The data $(\mathcal{F}_0, \nu_0)$, $(\mathcal{F}_1, \nu_1)$ and θ together define a morphism from $(\mathscr{C}^s \times \mathscr{I}, \mathcal{D}^s \circ p)$ to $(\mathscr{C}^t, \mathcal{D}^t)$. By Remark B.7, this induces a map from $\operatorname{hocolim}(\mathcal{D}^s \circ p) \cong (\operatorname{hocolim} \mathcal{D}^s) \times B\mathscr{I}$ to $\operatorname{hocolim} \mathcal{D}^t$. □

Let $\mathscr{C}$ be a small category and let $a \mapsto \mathcal{F}_a$ be a covariant functor from $\mathscr{C}$ to the category of sheaves on $\mathscr{X}$.

Lemma B.9. $|\operatorname{hocolim}_a \mathcal{F}_a| \simeq \operatorname{hocolim}_a |\mathcal{F}_a|$.

Proof. Definition 4.3 and Theorem 4.2 give $|\operatorname{hocolim}_a \mathcal{F}_a| \simeq B|\mathscr{C} \textstyle\int \mathcal{F}|$ and Propositions A.3, A.5 imply $B|\mathscr{C} \textstyle\int \mathcal{F}| \cong B(\mathscr{C} \textstyle\int |\mathcal{F}_\bullet|)$, where $|\mathcal{F}_\bullet|$ denotes the functor $a \mapsto |\mathcal{F}_a|$ from $\mathscr{C}$ to spaces. □

Corollary B.10. *Let* $\mathscr{C}$ *be a small category. Let*

$$a \mapsto \mathcal{E}_a \quad \textit{and} \quad a \mapsto \mathcal{E}'_a$$

be covariant functors from $\mathscr{C}$ *to the category of sheaves on* $\mathscr{X}$. *Let* $\nu = \{\nu_a\colon \mathcal{E}_a \to \mathcal{E}'_a\}$ *be a natural transformation such that every* $\nu_a\colon \mathcal{E}_a \to \mathcal{E}'_a$ *is a weak equivalence. Then the induced map* $\operatorname{hocolim}_a \mathcal{E}_a \to \operatorname{hocolim}_a \mathcal{E}'_a$ *is a weak equivalence* (*between sheaves on* $\mathscr{X}$). □

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