FINAL EXAM REVIEW-II

(1) Consider the system in polar coordinates:

$$\begin{cases} r' &= r - r^3 \\ \theta' &= 1 \end{cases}$$

Find the limits sets $\omega(x)$ and $\alpha(x)$ for all points $x \in \mathbf{R}^2$.

Find all periodic solutions.

(2) Formulate the Poincaré-Bendixon Theorem.

(a) Show that the system

$$\begin{cases} x' &= x(x^2 + y^2 - 3x - 1) - y \\ y' &= y(x^2 + y^2 - 3x - 1) + x \end{cases}$$

has at least one periodic solution.

(b) Show that the system

$$\begin{cases} x' &= -x(x^2 + y^2 - 3x - 1) + y \\ y' &= -y(x^2 + y^2 - 3x - 1) - x \end{cases}$$

has at least one periodic solution.

(3) Consider the system

$$\begin{cases} x' &= -y - x\sqrt{x^2 + y^2} \\ y' &= \quad x - y\sqrt{x^2 + y^2} \end{cases}$$

(a) Use polar coordinate system to find an explicit general solution.

(b) Use Lyapunov function of the type $V(x, y) = a^2x^2 + b^2y^2$ to show that the origin is an asymptotically stable critical point.

(c) Sketch few trajectories.

(4) Consider the Lorenz system:

$$\begin{cases} x' &= \sigma(y - x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{cases}$$

Here $\sigma, r, b > 0$ and $\sigma > b + 1$.

(a) Find critical points of this system.

(b) Use Lyapunov function $V(x, y) = x^2 + \sigma y^2 + \sigma z^2$ to show that the origin is an asymptotically stable critical point provided $0 < r < 1$.

(5) Consider the Predator-Prey System:

$$\begin{cases} x' &= x(a - by) \\ y' &= y(-c + dx) \end{cases}$$

Here $a, b, c, d > 0$.

(a) Find critical points of this system.

(b) Use Lyapunov function $V(x, y)$ of the type $F(x) + G(y)$ to show that one of the critical points is stable.