FINAL TEST REVIEW I

1. We define numbers a_n recursively:

$$a_0 = 1, \quad a_1 = 1; \quad \text{and} \quad a_n = 3a_{n-1} + 2a_{n-2}.$$

Compute $a_2, a_3, \dots, a_7$. Prove that all a_n are odd integers.

2. Prove that $n^5 - n$ is divisible by 10 for all positive integers n.

3. Let p be a prime. Prove that $n^p - n$ is divisible by p for any integer n.

4. Let m, n be integers and $n > 0$. Show that $\gcd(m,n) = \gcd(n, m \pmod{n})$.

5. Let $p = 79$ and $q = 113$. Find integers t and s such that $79t + 113s = 1$. Use this result to find $[79]^{-1}$ in $\mathbf{Z}_{113}$.

6. Find $[2011]^{-1}$ in $\mathbf{Z}_{2016}$.

7. Solve the following equations

(a) $2000x \equiv 21 \pmod{643}$
(b) $643y \equiv 13 \pmod{2000}$
(c) $1647z \equiv 92 \pmod{788}$
(d) $788w \equiv 24 \pmod{1647}$

8. Find the last two digits of the number 2016^{2016}.

9. Calculate $\phi(2016)$, $\phi(\phi(2016))$, $\phi(\phi(\phi(2016)))$ and $\phi(\phi(\phi(\phi(2016))))$, where ϕ is the Euler function.

10. Let $A \subset \Sigma^*$ be a language, $\Sigma = \{0, 1\}$. Provide a recursive definitions for the following languages:

(a) $x \in A$ if and only if x has odd number of 1's;
(b) $x \in A$ if and only if x has even number of 0's;
(c) $x \in A$ if and only if x has odd number of 1's and even number of 0's;
(d) $x \in A$ if and only if x has odd number of 0's or odd number of 1's.

11. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) Determine the number of reflexive relations on A.
(b) Determine the number of symmetric relations on A.
(c) Determine the number of reflexive and symmetric relations on A.
(d) Determine the number of antisymmetric relations on A.
(e) Determine the number of reflexive and antisymmetric relations on A.

12. Design a finite state machine $M = (S, O, \nu, \omega)$, where $S = O = \{0, 1\}$, which recognized a patern "110111" in a binary string.

13. Design a finite state machine $M = (S, O, \nu, \omega)$, where $S = O = \{0, 1\}$, which recognized a patern "110111" in a binary string only when the zero occurs at the position which is multiple of 3.

14. Let A be a set of all divisors of $18,000$. Find the number of pairs (a, b) such that a divides b, and $a, b \in A$.

15. Let A and B be finite sets with $|A| = 11$, $|B| = 6$. How many onto functions $f : A \to B$ are there?

16. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be a decomposition of n through primes. We assume that $p_1 < p_2 < \cdots < p_k$.

(a) How many divisors d of n are there?

(b) For two divisors d, d' of n, we write $d \le d'$ (or $(d, d') \in \mathcal{R}$) iff d divides d'. Find the size of the set $|\mathcal{R}|$.

17. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Determine the number of equivalence relations on A.

18. Let A be a set with n elements, $\mathcal{R}$ be a relation on A, and $M = M(\mathcal{R})$ denotes the $(0,1)$-matrix corresponding to the relation $\mathcal{R}$.

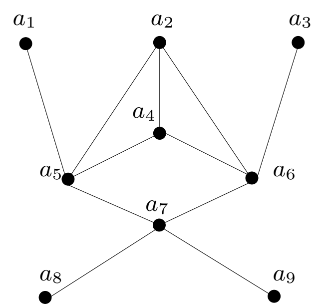
(i) Prove that $\mathcal{R}$ is reflexive if and only if $I_n \le M$.

(ii) Prove that $\mathcal{R}$ is symmetric if and only if $M = M^T$.

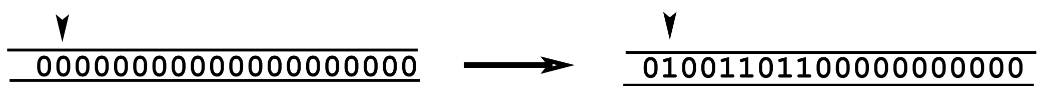
(iii) Prove that $\mathcal{R}$ is transitive if and only if $M^2 \le M$.

19. Let A be a finite poset. Prove that A has a maximal element.

20. For a poset $A = \{a_1, \ldots, a_9\}$, the Hasse diagram is shown below. Topologically sort this Hasse diagram.



21. Design a Turing machine which starts with a blank tape of zeros and halts after it produces the pattern 10011011:



22. Let A be a finite set. Prove that there is one-to-one correspondence between equivalence relations on A and partitions of A.

23. Let $A = \{a_1, \ldots, a_m\}$. Prove that there are

$$\sum_{k=1}^{m} S(m,k) = \sum_{k=1}^{m} \left(\frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m \right)$$

partitions of $A = \{a_1, \ldots, a_m\}$.

24. Prove that an element $k \in \mathbf{Z}/n$ is a unit if and only if $\gcd(k,n) = 1$. How many units are there in $\mathbf{Z}_{2016}$?

25. State and prove Fermat's Little Theorem.

26. Compute 2011^{2011} mod 31.