Summary on Lecture 9, April 13th, 2016

Equivalence relations and partitions: again.

Let A be a set. A family of subsets $\{A_i\}_{i\in I}$, $A_i \subseteq A$, is called a partition if

$$A = \bigcup_{i\in I} A_i, \quad A_i \cap A_{i'} = \emptyset \text{ if } i \neq i'.$$

Important example. Let $n \in \mathbf{Z}_+$ be a positive integer. We define an equivalence relation on $\mathbf{Z}$ as follows: $m \sim m'$ iff $m - m'$ is divisible by n. Then we have n different classes of equivalent integers:

$$\begin{array}{ccl} \mathbf{0} & := & \{0, \pm n, \pm 2\cdot n, \ldots\}, \\ \mathbf{1} & := & \{1, 1\pm n, 1\pm 2\cdot n, \ldots\}, \\ \mathbf{2} & := & \{2, 2\pm n, 2\pm 2\cdot n, \ldots\}, \\ \cdots & & \cdots\cdots\cdots \\ \mathbf{n}-1 & := & \{n-1, n-1\pm n, n-1\pm 2\cdot n, \ldots\}. \end{array}$$

We obtain that $\mathbf{Z} = \bigcup_{i=0}^{n-1} \mathbf{i}$, and clearly the sets $\mathbf{i}$ and $\mathbf{i}'$ do not intersect if $i \neq i'$. The set of equivalent classes $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}-1\}$ is denoted by $\mathbf{Z}/n$. There are well-defined sum and product operations on $\mathbf{Z}/n$:

$$\mathbf{i} + \mathbf{i}' \quad \text{and} \quad \mathbf{i}\cdot\mathbf{i}'$$

Here are the addition and multiplication tables in $\mathbf{Z}/5$:

+	**0**	**1**	**2**	**3**	**4**
0	**0**	**1**	**2**	**3**	**3**
1	**1**	**2**	**3**	**4**	**0**
2	**2**	**3**	**4**	**0**	**1**
3	**3**	**4**	**0**	**1**	**2**
4	**4**	**0**	**1**	**2**	**3**

×	**0**	**1**	**2**	**3**	**4**
0	**0**	**0**	**0**	**0**	**0**
1	**0**	**1**	**2**	**3**	**4**
2	**0**	**2**	**4**	**1**	**3**
3	**0**	**3**	**1**	**4**	**2**
4	**0**	**4**	**3**	**2**	**1**

Next, we have the following addition and multiplication tables in $\mathbf{Z}/6$:

+	**0**	**1**	**2**	**3**	**4**	**5**
0	**0**	**1**	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**	**5**	**0**
2	**2**	**3**	**4**	**5**	**0**	**1**
3	**3**	**4**	**5**	**0**	**1**	**2**
4	**4**	**5**	**0**	**1**	**2**	**3**
5	**5**	**0**	**1**	**2**	**3**	**4**

×	**0**	**1**	**2**	**3**	**4**	**5**
0	**0**	**0**	**0**	**0**	**0**	**0**
1	**0**	**1**	**2**	**3**	**4**	**5**
2	**0**	**2**	**4**	**0**	**2**	**4**
3	**0**	**3**	**0**	**3**	**0**	**3**
4	**0**	**4**	**2**	**0**	**4**	**2**
5	**0**	**5**	**4**	**3**	**2**	**1**

We notice that $\mathbf{2}\cdot\mathbf{3} = \mathbf{0}$, $\mathbf{4}\cdot\mathbf{3} = \mathbf{0}$, and $\mathbf{3}\cdot\mathbf{3} = \mathbf{3}$.

Let $\mathcal{R} \subset A\times A$ be an equivalence relation. For each element $x \in A$ we define a subset

$$[x] = \{\ y \in A \mid (x,y) \in \mathcal{R}\ \}$$

We notice that either $[x] = [x']$ or $[x] \cap [x'] = \emptyset$. Indeed, assume that $[x]\cap[x'] \neq \emptyset$, and $z \in [x]\cap[x']$. Then $(x,z) \in \mathcal{R}$ and $(x',z) \in \mathcal{R}$, and this implies that $(x,x') \in \mathcal{R}$, and thus $[x] = [x']$. We obtain that the family of sets $\{[x]\}$ is a partition of A.

Now let $\{A_i\}_{i\in I}$ be a partition of A. Then we define a relation $\mathcal{R} \subset A\times A$ as follows:

$$(x,x') \in \mathcal{R} \text{ iff there exists } \quad i \in I \quad \text{such that} \quad x, x' \in A_i.$$

It is easy to check that $\mathcal{R} \subset A\times A$ is an equivalence relation.

Theorem 1. Let A be a set. Then there is one-to-one correspondence between equivalence relations on A and partitions of A.

Now we would like to count all possible partitions of a finite set $A = \{a_1, \ldots, a_m\}$. We'll say that a partition $A = \bigcup_{i=1}^k A_i$ has a size k. Clearly, $1 \leq k \leq m$. We fix such k and count how many partitions of size k are there.

To get started, we can count how many onto maps are there $f : A \to B$, where $B = \{b_1, \dots, b_k\}$. Then we can think of b_i as a box to collect elements for A_i, thus we should forget the order of those boxes.

We denote by S the set of all maps $f : A \to B$. Since $|A| = m$, $|B| = k$, we conclude that $|S| = k^m$. Now for each $i = 1, 2, \dots, k$, we denote by S_i the following set of maps:

$$S_i = \{ \ f : B \to A \mid b_i \notin f(A) \ \}$$

Then it is clear that $|S_i| = (k-1)^m$. Then we identify the set of all onto maps $f : A \to B$ with the set

$$S \setminus (S_1 \cup \dots \cup S_k).$$

By using the inclusion-exclusion principle, we obtain

$$|S \setminus (S_1 \cup \dots \cup S_k)| = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m$$

onto functions are there $f : A \to B$. We divide by $k!$ to obtain the Stirling number

$$S(m,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m.$$

Now we sum up partitions of A of all sizes. We obtain that there are

$$\sum_{k=1}^{m} S(m,k) = \sum_{k=1}^{m} \left(\frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m \right)$$

partitions of $A = \{a_1, \dots, a_m\}$.