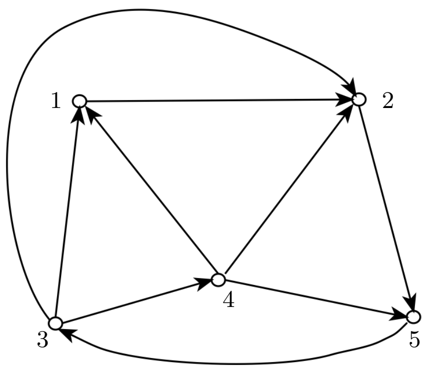
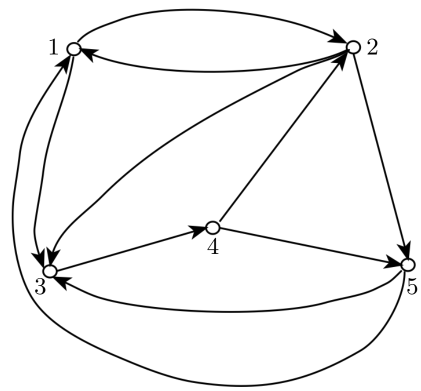
Summary on Lecture 7, April 8th, 2016

Zero-one matrices and graphs.

Let $G = (V, E)$ be a directed graph (digraph), where $|V| = n$. Then a vertex $e = (v, v') \in E$ is an ordered pair of vertices. Thus we can describe vertices as a binary relation on the set of vertices. Let us consider the following directed graphs:



(a) Digraph G_1



(b) Digraph G_2

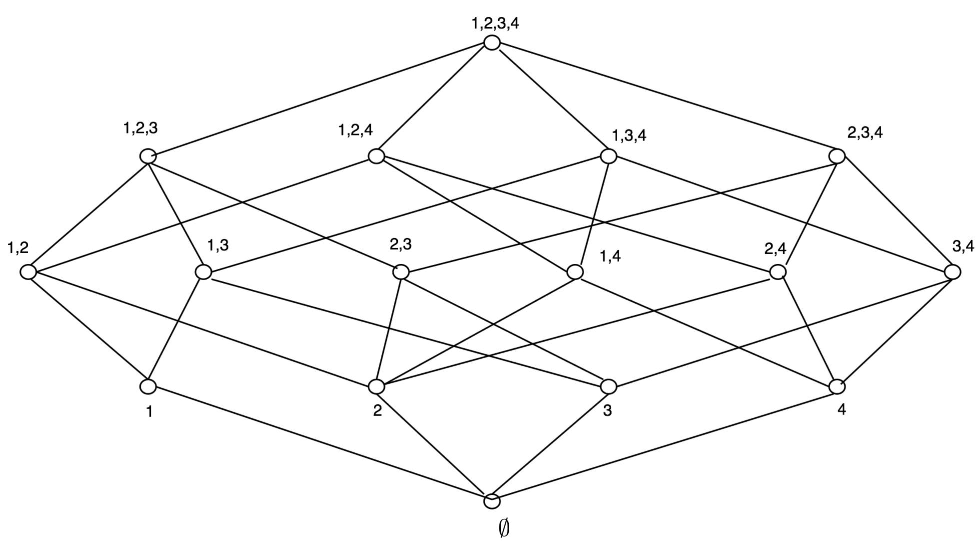
Here we have the adjacency matrices $M(G_1)$ and $M(G_2)$ of these digraphs:

$$M(G_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad M(G_2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

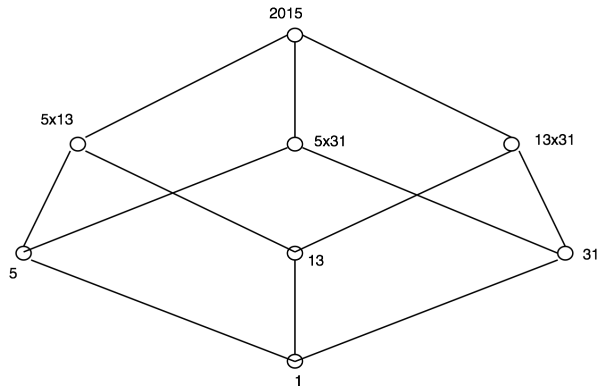
Partial order and Hasse diagrams.

Let A be a set. We recall that a *partial order on* A is a binary relation $\mathcal{R} \subset A \times A$ which is reflexive, antisymmetric and transitive. A useful tool to work with a partial order is a *Hasse diagram*. We give several examples.

Examples. (a) Let S be a set, and $\mathcal{P}(S)$ be a set of all subsets of S. Then we define the relation: for $A, B \in \mathcal{P}(S)$, $A \leq B$ iff $A \subseteq B$. Here is a *Hasse diagram* of this partial order if $S = \{1, 2, 3, 4\}$.



(b) Let $n = p_1^{e_1} \cdots p_k^{e_k}$ be a prime decomposition of a positive integer n. Let $D(n)$ be the set of all divisors of n. Then every $d \in D(n)$ has a form $d = p_1^{a_1} \cdots p_k^{a_k}$, where $0 \leq a_i \leq e_i$ for each $i = 1, \ldots, k$. We already have considered the following partial order on $D(n)$: $d \leq d'$ iff d is a divisor of d'. Below is a Hasse diagram for this partial order if $n = 2015$.

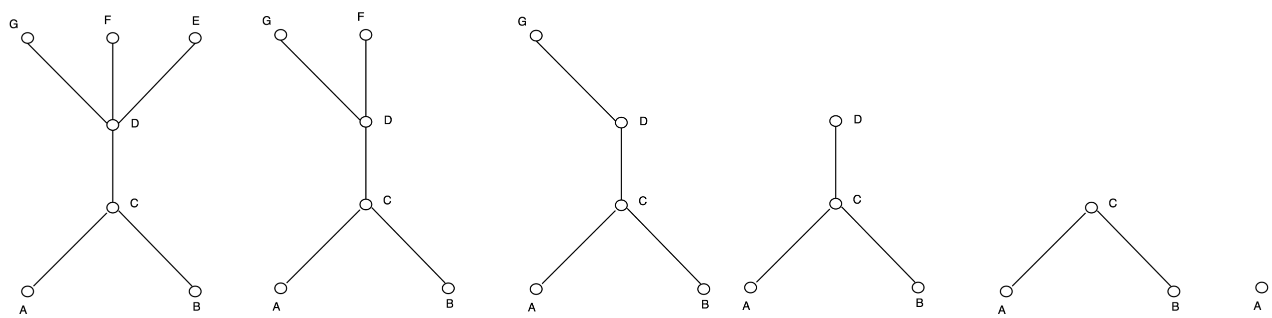


Let $\mathcal{R}$ be a partial order on A. We say that $\mathcal{R}$ is a *linear (or total) order* on A if for any $x, y \in A$ either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$.

Example. Let $A = \{1, 2, 2^2, \ldots, 2^k\}$, and $(x \leq y)$ iff x is a divisor of y, i.e., $x|y$. This is a linear order on A.

There are some practical applications of these concepts.

Example. Assume we would like to manufacture a product X (say, a toy). In order to do that there are several operations we have to perform according to the following Hasse diagram:



Nevertheless we have to organize the production in linear order since those operations could not be done at the same time. We select (from right to left) a "highest" leaf in that diagram, and we delete it. Then we recur. We obtain the following linear order: $A < B < C < D < G < F < E$. This new linear order provides a "linear" process to manufacture our product.

Definition. Let A a poset (i.e., a partial ordered set). We say that $x \in A$ is maximal if $x \leq a$ implies $x = a$. Similarly, $y \in A$ is minimal if $a \leq y$ implies $a = y$.

Theorem 1. Let A be a finite poset. Then there exists a maximal (minimal) element in A.

Exercise. Prove Theorem 1.

Definition. Let A a poset. An element $x \in A$ is a greatest element if $a \leq x$ for all $a \in A$. Similarly, an element $y \in A$ is a least element if $y \leq a$ for all $a \in A$.

Theorem 2. Let A be a poset. Assume there exists a greatest (least) element in A. Then it is unique.

Exercise. Prove Theorem 2.