Summary on Lecture 5, April 4th, 2016

Equivalence and partial order relations

Let A, B be sets (which live, as usual, in some "universal set"). Recall that a subset $\mathcal{R} \subset A \times B$ is called a *binary relation*.

Example 1. Let $A = B = \mathbf{Z}$, and $n \in \mathbf{Z}$. Then $(k, \ell) \in \mathcal{R}$ if and only if $k - \ell \equiv 0 \mod n$.

Example 2. Let $A = B = \mathbf{Z}$. Then $(k, \ell) \in \mathcal{R}$ if and only if $k \leq \ell$.

Now we let $\mathcal{R} \subset A \times A$ be a *binary relation on* A, i.e., when $A = B$.

Definition. We say that a binary relation $\mathcal{R}$ on A is an *equivalence relation* if it satisfies the following properties:
(R) $(x, x) \in \mathcal{R}$ for each $a \in A$ (Reflexivity);
(S) if $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$ (Symmetry);
(T) if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(y, z) \in \mathcal{R}$ (Transitivity).

Exercise. Check that the relation $\mathcal{R}$ from Example 1 is an equivalence relation, and that is not true for the relation from Example 2.

Definition. We say that a binary relation $\mathcal{R}$ on A is an *partial order on* A if it satisfies the following properties:
(R) $(x, x) \in \mathcal{R}$ for each $a \in A$ (Reflexivity);
(A) if $(x, y) \in \mathcal{R}$, and $(y, x) \in \mathcal{R}$, then $x = y$ (Antisymmetry);
(T) if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(y, z) \in \mathcal{R}$. (Transitivity).

Remark. The usual relation partial order "$\leq$" on real numbers satisfies all the properties (R), (A), (T).

In order to understand well the above realtions, we will do some counting. Let A be a finite set, $|A| = n$.
(R) Let $\mathcal{R}$ be a reflexive relation on A. Then $(a, a) \in \mathcal{R}$ for all $a \in A$. Thus $\mathcal{R}$ contains at least the diagonal $\{(a_1, a_1), \ldots, (a_n, a_n)\}$, and $\mathcal{R}$ may contain any subset from $A \times A \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\}$. Thus we have 2^{n^2-n} reflexive relations on A.
(S) Let $\mathcal{R}$ be a symmetric relation on A. To count how many such relation we have, we notice that the difference $A \times A \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\}$ consists of pairs (a_i, a_j) with $i \neq j$. Then if $(a_i, a_j) \in \mathcal{R}$, then $(a_j, a_i) \in \mathcal{R}$, so it is enough to count pairs (a_i, a_j) with $i \leq j$. We obtain $2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$ symmetric relations.

Example 3. Here is an interesting example of partial order. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be a decomposition of n through primes. We assume that $p_1 < p_2 < \cdots < p_k$. Then every divisor d of n has a form $d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $0 \leq a_i \leq e_i$ for each $i = 1, 2, \ldots, k$. Thus n has $\prod_{i=1}^{k}(e_i + 1)$ divisors. Then for two divisors d, d' we write $d \leq d'$ (or $(d, d') \in \mathcal{R}$) iff d divides d'. Let

$$d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad d' = p_1^{a'_1} p_2^{a'_2} \cdots p_k^{a'_k}$$

be two divisors of n. Then d divides d' iff $0 \leq a_i \leq a'_i \leq e_i$ for each $i = 1, 2, \ldots, k$. Consider just one index i: we can use the problem of counting number of ways to place 2 objects to $e_i + 1$ boxes. We obtain $\binom{e_i+1+2-1}{2} = \binom{e_i+2}{2}$ pairs (a_i, a'_i) satisfying $0 \leq a_i \leq a'_i \leq e_i$. We obtain:

$$|\mathcal{R}| = \prod_{i=1}^{k} \binom{e_i + 2}{2}.$$

More on relations: matrices and graphs.

Definition. Let A, B, C be sets, and $\mathcal{R}_1 \subset A \times B$, $\mathcal{R}_2 \subset B \times C$ be two binary relations. A composite relation $\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2$ is defined as the set

$$\mathcal{R} := \{ \ (a, c) \mid \text{there exists } b \in B \text{ such that } (a, b) \in \mathcal{R}_1 \text{ and } (b, c) \in \mathcal{R}_2 \ \}.$$

Example. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{w, x, y, z\}$, $C = \{a, b, c\}$. The relations $\mathcal{R}_1 \subset A \times B$, $\mathcal{R}_2 \subset B \times C$ are given as follows:

$\mathcal{R}_1 =$

	w	x	y	z
1	1	1	0	0
2	0	0	1	0
3	1	0	1	0
4	0	1	0	1
5	1	0	1	0

$\mathcal{R}_2 =$

	a	b	c
w	1	0	0
x	0	1	0
y	0	0	0
z	0	1	1

Here we put 1's for all pairs (a, b) and (b, c) such that $(a, b) \in \mathcal{R}_1$ and $(b, c) \in \mathcal{R}_2$; otherwise we put zeros. We obtain the matrices

$$M(\mathcal{R}_1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad M(\mathcal{R}_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

which are called *zero-one matrices corresponding to the realtions* $\mathcal{R}_1$ *and* $\mathcal{R}_2$. We notice that the relation $\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2$ has the following matrix:

$$M(\mathcal{R}_1 \circ \mathcal{R}_2) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Here we use a standard multiplication of matrices, plus we used the following rules: $0 \cdot 0 = 0$, $0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$, and $1 + 1 = 1$. The last rule is designed specifically for zero-one matrices.

Theorem 1. Let Let A, B, C, D be sets, and $\mathcal{R}_1 \subset A \times B$, $\mathcal{R}_2 \subset B \times C$, $\mathcal{R}_3 \subset C \times D$ be binary relations. Then

$$(\mathcal{R}_1 \circ \mathcal{R}_1) \circ \mathcal{R}_3 = \mathcal{R}_1 \circ (\mathcal{R}_1 \circ \mathcal{R}_3).$$

Exercise. Prove Theorem 1.

Let $\mathcal{R} \subset A \times A$ be a binary relation. Then a power $\mathcal{R}^\ell$ is defined recursively: $\mathcal{R}^1 := \mathcal{R}$, $\mathcal{R}^{\ell+1} := \mathcal{R} \circ \mathcal{R}^\ell$.

Examples. (1) Let $A = \{1, 2, 3, 4, 5\}$, and

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad M(\mathcal{R}^2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M(\mathcal{R}^3) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can notice that $M(\mathcal{R}^{2k+1}) = M(\mathcal{R})$ and $M(\mathcal{R}^{2k}) = M(\mathcal{R}^2)$. Thus $\mathcal{R}^{2k+1} = \mathcal{R}$ and $\mathcal{R}^{2k} = \mathcal{R}^2$.

(2) Let $A = \{1, 2, 3, 4\}$, and

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(\mathcal{R}^2) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(\mathcal{R}^3) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(\mathcal{R}^4) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain that $\mathcal{R}^\ell = \emptyset$ for $\ell \geq 4$.

Exercise. Let A be a finite set with $|A| = n$, and $\mathcal{R} \subset A \times A$ be a relation. Prove the following properties of the zero-one matrix $M(\mathcal{R})$:

(a) $M(\mathcal{R}) = 0$ (the matrix of all 0's) if and only if $\mathcal{R} = \emptyset$.
(b) $M(\mathcal{R}) = 1$ (the matrix of all 1's) if and only if $\mathcal{R} = A \times A$.
(c) $M(\mathcal{R}^\ell) = M^\ell(\mathcal{R})$ for all $\ell \geq 1$.

Let $M = \{m_{ij}\}$, $M' = \{m'_{ij}\}$ be two zero-one matrices of the same size. We say that $M \leq M'$ if and only if $m_{ij} \leq m'_{ij}$ for all indices i, j. We denote by I_n the identity matrix of the size n. Also if M is a matrix, we denote by M^T its transpose. Finally, if $M = \{m_{ij}\}$, $M' = \{m'_{ij}\}$ are two zero-one matrices of the same size, we define the matrix $M \cap M' = \{m''_{ij}\}$ as

$$m''_{ij} = \begin{cases} 1 & \text{if} \quad m_{ij} = m'_{ij} = 1 \\ 0 & \text{else} \end{cases}$$

Theorem 2. Let A be a finite set with $|A| = n$, and $\mathcal{R} \subset A \times A$ be a relation. Then

(a) $\mathcal{R}$ is reflexive if and only if $I_n \leq M(\mathcal{R})$;

(b) $\mathcal{R}$ is symmetric if and only if $M(\mathcal{R})^T = M(\mathcal{R})$;

(c) $\mathcal{R}$ is transitive if and only if $M(\mathcal{R})^2 \leq M(\mathcal{R})$;

(d) $\mathcal{R}$ is symmetric if and only if $M(\mathcal{R}) \cap M(\mathcal{R})^T \leq I_n$.

Exercise. Prove any three of the statements (a), (b), (c) or (d).