Summary on Lecture 10, April 15th, 2016

Equivalence relations and partitions: counting number of partitions.

Let A be a set. A family of subsets $\{A_i\}_{i\in I}$, $A_i \subseteq A$, is called a partition if

$$A = \bigcup_{i\in I} A_i, \quad A_i \cap A_{i'} = \emptyset \text{ if } i \neq i'.$$

Let $\mathcal{R} \subset A \times A$ be an equivalence relation. For each element $x \in A$ we define a subset

$$[x] = \{ \ y \in A \mid (x,y) \in \mathcal{R} \ \}$$

We notice that either $[x] = [x']$ or $[x] \cap [x'] = \emptyset$. Indeed, assume that $[x] \cap [x'] \neq \emptyset$, and $z \in [x] \cap [x']$. Then $(x,z) \in \mathcal{R}$ and $(x',z) \in \mathcal{R}$, and this implies that $(x,x') \in \mathcal{R}$, and thus $[x] = [x']$. We obtain that the family of sets $\{[x]\}$ is a partition of A.

Now let $\{A_i\}_{i\in I}$ be a partition of A. Then we define a relation $\mathcal{R} \subset A \times A$ as follows:

$$(x,x') \in \mathcal{R} \text{ iff there exists } \ i \in I \ \text{ such that } \ x,x' \in A_i.$$

It is easy to check that $\mathcal{R} \subset A \times A$ is an equivalence relation.

Theorem 1. Let A be a set. Then there is one-to-one correspondence between equivalence relations on A and partitions of A.

Now we would like to count all possible partitions of a finite set $A = \{a_1, \ldots, a_m\}$. We'll say that a partition $A = \bigcup_{i=1}^k A_i$ has a size k. Clearly, $1 \leq k \leq m$. We fix such k and count how many partitions of size k are there.

To get started, we can count how many onto maps are there $f : A \to B$, where $B = \{b_1, \ldots, b_k\}$. Then we can think of b_i as a box to collect elements for A_i, thus we should forget the order of those boxes.

We denote by S the set of all maps $f : A \to B$. Since $|A| = m$, $|B| = k$, we conclude that $|S| = k^m$. Now for each $i = 1, 2, \ldots, k$, we denote by S_i the following set of maps:

$$S_i = \{ \ f : B \to A \mid b_i \notin f(A) \ \}$$

Then it is clear that $|S_i| = (k-1)^m$. Then we identify the set of all onto maps $f : A \to B$ with the set

$$S \setminus (S_1 \cup \cdots \cup S_k).$$

By using the inclusion-exclusion principle, we obtain

$$|S \setminus (S_1 \cup \cdots \cup S_k)| = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m$$

onto functions are there $f : A \to B$. We divide by $k!$ to obtain the Stirling number

$$S(m,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m.$$

Now we sum up partitions of A of all sizes. We obtain that there are

$$\sum_{k=1}^{m} S(m,k) = \sum_{k=1}^{m} \left(\frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m \right)$$

partitions of $A = \{a_1, \ldots, a_m\}$.