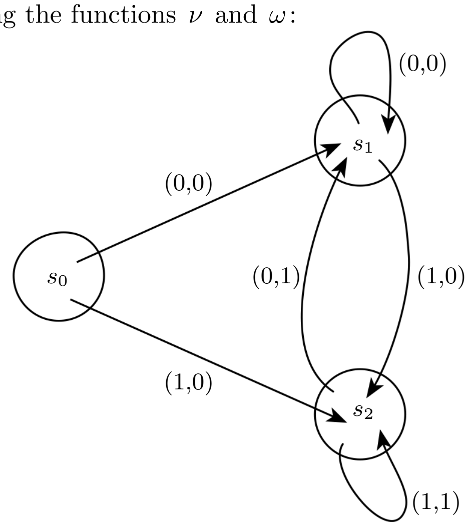
Summary on Lecture 4, April 3d, 2015

Finite State Machines: More examples.

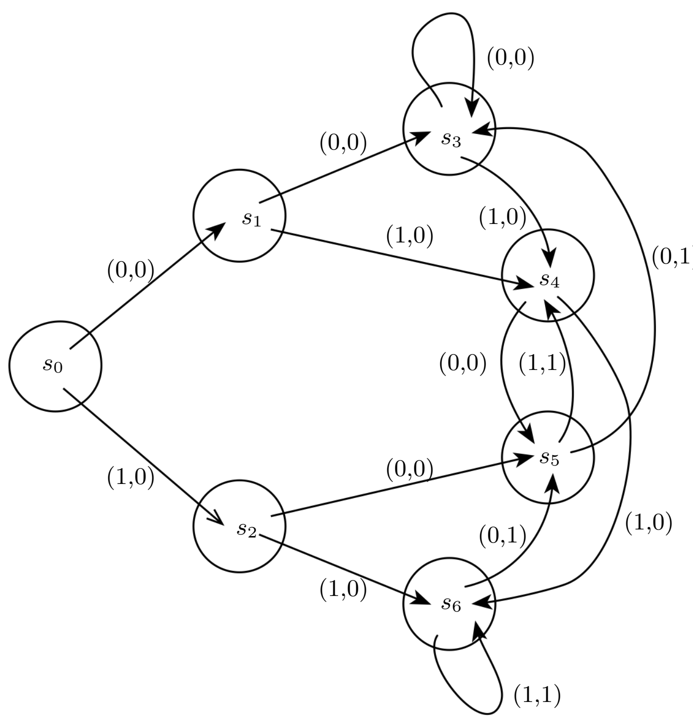
(7) Delay machine. Now we describe a finite state machine which delays the sequence by putting first k zeros. For example, if $k = 1$, the input sequence 11110111110111010110... gives the following output

11110111110111010110...
011110111110111010110...

Here is the diagram describing the functions ν and ω:



The case $k = 2$ is essentially more complicated since the machine has to remember two previous digits. Here the input sequence 11110111110111010110... gives the output 0011110111110111010110... Here is the diagram describing the functions ν and ω:



We notice that the states s_0, s_1, s_2 have only 0 as an output, and the states s_3, s_4, s_5, s_6 "remember" the prior inputs 00, 10, 10, 11 respectively.

Exercise. Construct a delay machine with $k = 3$.

Equivalence and partial order relations

Let A, B be sets (which live, as usual, in some "universal set"). Recall that a subset $\mathcal{R} \subset A \times B$ is called a *binary relation.*

Example 1. Let $A = B = \mathbf{Z}$, and $n \in \mathbf{Z}$. Then $(k, \ell) \in \mathcal{R}$ if and only if $k - \ell \equiv 0 \mod n$.

Example 2. Let $A = B = \mathbf{Z}$. Then $(k, \ell) \in \mathcal{R}$ if and only if $k \leq \ell$.

Now we let $\mathcal{R} \subset A \times A$ be a *binary relation on* A, i.e., when $A = B$.

Definition. We say that a binary relation $\mathcal{R}$ on A is an *equivalence relation* if it satisfies the following properties:
(R) $(x, x) \in \mathcal{R}$ for each $a \in A$ (Reflexivity);
(S) if $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$ (Symmetry);
(T) if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(y, z) \in \mathcal{R}$ (Transitivity).

Exercise. Check that the relation $\mathcal{R}$ from Example 1 is an equivalence relation, and that is not true for the relation from Example 2.

Definition. We say that a binary relation $\mathcal{R}$ on A is an *partial order on* A if it satisfies the following properties:
(R) $(x, x) \in \mathcal{R}$ for each $a \in A$ (Reflexivity);
(A) if $(x, y) \in \mathcal{R}$, and $(y, x) \in \mathcal{R}$, then $x = y$ (Antisymmetry);
(T) if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(y, z) \in \mathcal{R}$. (Transitivity).

Remark. The usual relation partial order "$\leq$" on real numbers satisfies all the properties (R), (A), (T).

In order to understand well the above realtions, we will do some counting. Let A be a finite set, $|A| = n$.

(R) Let $\mathcal{R}$ be a reflexive relation on A. Then $(a, a) \in \mathcal{R}$ for all $a \in A$. Thus $\mathcal{R}$ contains at least the diagonal $\{(a_1, a_1), \ldots, (a_n, a_n)\}$, and $\mathcal{R}$ may contain any subset from $A \times A \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\}$. Thus we have $2^{n^2 - n}$ reflexive relations on A.

(S) Let $\mathcal{R}$ be a symmetric relation on A. To count how many such relation we have, we notice that the difference $A \times A \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\}$ consists of pairs (a_i, a_j) with $i \neq j$. Then if $(a_i, a_j) \in \mathcal{R}$, then $(a_j, a_i) \in \mathcal{R}$, so it is enough to count pairs (a_i, a_j) with $i \leq j$. We obtain $2^n \cdot 2^{\frac{n^2 - n}{2}} = 2^{\frac{n^2 + n}{2}}$ symmetric relations.

Example 3. Here is an interesting example of partial order. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be a decomposition of n through primes. We assume that $p_1 < p_2 < \cdots < p_k$. Then every divisor d of n has a form $d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $0 \leq a_i \leq e_i$ for each $i = 1, 2, \ldots, k$. Thus n has $\displaystyle\prod_{i=1}^{k} (e_i + 1)$ divisors. Then for two divisors d, d' we write $d \leq d'$ (or $(d, d') \in \mathcal{R}$) iff d divides d'. Let

$$d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad d' = p_1^{a_1'} p_2^{a_2'} \cdots p_k^{a_k'}$$

be two divisors of n. Then d divides d' iff $0 \leq a_i \leq a_i' \leq e_i$ for each $i = 1, 2, \ldots, k$. Consider just one index i: we can use the problem of counting number of ways to place 2 objects to $e_i + 1$ boxes. We obtain $\binom{e_i + 1 + 2 - 1}{2} = \binom{e_i + 2}{2}$ pairs (a_i, a_i') satisfying $0 \leq a_i \leq a_i' \leq e_i$. We obtain:

$$|\mathcal{R}| = \prod_{i=1}^{k} \binom{e_i + 2}{2}.$$