Finite State Machines: More examples.

(7) Delay machine. Now we describe a finite state machine which delays the sequence by putting first \( k \) zeros. For example, if \( k = 1 \), the input sequence 11110111110110110... gives the following output

\[
\begin{align*}
11110111110110110... \\
01111011110110110... \\
\end{align*}
\]

Here is the diagram describing the functions \( \nu \) and \( \omega \):

![Diagram](image)

The case \( k = 2 \) is essentially more complicated since the machine has to remember two previous digits. Here the input sequence 11110111110110110... gives the output 0011110111110110110... Here is the diagram describing the functions \( \nu \) and \( \omega \):

![Diagram](image)
We notice that the states $s_0, s_1, s_2$ have only 0 as an output, and the states $s_3, s_4, s_5, s_6$ “remember” the prior inputs 00, 10, 10, 11 respectively.

**Exercise.** Construct a delay machine with $k = 3$.

**Equivalence and partial order relations**

Let $A, B$ be sets (which live, as usual, in some “universal set”). Recall that a subset $\mathcal{R} \subset A \times B$ is called a binary relation.

**Example 1.** Let $A = B = \mathbb{Z}$, and $n \in \mathbb{Z}$. Then $(k, \ell) \in \mathcal{R}$ if and only if $k - \ell \equiv 0 \mod n$.

**Example 2.** Let $A = B = \mathbb{Z}$. Then $(k, \ell) \in \mathcal{R}$ if and only if $k \leq \ell$.

Now we let $\mathcal{R} \subset A \times A$ be a binary relation on $A$, i.e., when $A = B$.

**Definition.** We say that a binary relation $\mathcal{R}$ on $A$ is an equivalence relation if it satisfies the following properties:

(R) $(x, x) \in \mathcal{R}$ for each $a \in A$ (Reflexivity);
(S) if $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$ (Symmetry);
(T) if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(y, z) \in \mathcal{R}$ (Transitivity).

**Exercise.** Check that the relation $\mathcal{R}$ from Example 1 is an equivalence relation, and that is not true for the relation from Example 2.

**Definition.** We say that a binary relation $\mathcal{R}$ on $A$ is a partial order on $A$ if it satisfies the following properties:

(R) $(x, x) \in \mathcal{R}$ for each $a \in A$ (Reflexivity);
(A) if $(x, y) \in \mathcal{R}$, and $(y, x) \in \mathcal{R}$, then $x = y$ (Antisymmetry);
(T) if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(y, z) \in \mathcal{R}$ (Transitivity).

**Remark.** The usual relation partial order “$\leq$” on real numbers satisfies all the properties (R), (A), (T).

In order to understand well the above relations, we will do some counting. Let $A$ be a finite set, $|A| = n$.

(R) Let $\mathcal{R}$ be a reflexive relation on $A$. Then $(a, a) \in \mathcal{R}$ for all $a \in A$. Thus $\mathcal{R}$ contains at least the diagonal $\{(a_1, a_1), \ldots, (a_n, a_n)\}$, and $\mathcal{R}$ may contain any subset from $A \times A \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\}$. Thus we have $2^{n^2-n}$ reflexive relations on $A$.

(S) Let $\mathcal{R}$ be a symmetric relation on $A$. To count how many such relation we have, we notice that the difference $A \times A \setminus \{(a_1, a_1), \ldots, (a_n, a_n)\}$ consists of pairs $(a_i, a_j)$ with $i \neq j$. Then if $(a_j, a_j) \in \mathcal{R}$, then $(a_j, a_i) \in \mathcal{R}$, so it is enough to count pairs $(a_i, a_j)$ with $i \leq j$. We obtain $2^n \cdot 2^{n^2-n} = 2^{n^2-n}$ symmetric relations.

**Example 3.** Here is an interesting example of partial order. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be a decomposition of $n$ through primes. We assume that $p_1 < p_2 < \cdots < p_k$. Then every divisor $d$ of $n$ has a form $d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $0 \leq a_i \leq e_i$ for each $i = 1, 2, \ldots, k$. Thus $n$ has $\prod_{i=1}^{k} (e_i + 1)$ divisors. Then for two divisors $d, d'$ we write $d \leq d'$ (or $(d, d') \in \mathcal{R}$) iff $d$ divides $d'$. Let $d = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $d' = p_1^{a'_1} p_2^{a'_2} \cdots p_k^{a'_k}$ be two divisors of $n$. Then $d$ divides $d'$ iff $0 \leq a_i \leq a'_i \leq e_i$ for each $i = 1, 2, \ldots, k$. Consider just one index $i$: we can use the problem of counting number of ways to place 2 objects to $e_i + 1$ boxes. We obtain $\binom{e_i + 1 + 2 - 1}{2} = \binom{e_i + 2}{2}$ pairs $(a_i, a'_i)$ satisfying $0 \leq a_i \leq a'_i \leq e_i$. We obtain:

$$|\mathcal{R}| = \prod_{i=1}^{k} \binom{e_i + 2}{2}.$$