Summary on Lecture 18, May 15th, 2015

## Turing Machines: More Examples.

**Example 4.** Let us design a Turing Machine to compute the function $f(n) = (n, n)$ for a nonnegative integer $n$, i.e., we need a machine which performs as follows:

$$0\underbrace{\overset{\downarrow}{1}1\ \ldots\ 11}_{n}0 \;\mapsto\; 0\underbrace{\overset{\downarrow}{1}1\ \ldots\ 11}_{n}0\underbrace{11\ \ldots\ 11}_{n}$$

We will do this in two steps.

**Step 1.** Here we will design a Turing Machine which does the following:

$$0\underbrace{\overset{\downarrow}{1}1\ \ldots\ 11}_{n}0 \;\mapsto\; 0\underbrace{11\ \ldots\ 11}_{n}\overset{\downarrow}{0}\underbrace{11\ \ldots\ 11}_{n}$$

It is difficult to do directly. We'll split this up further by describing a Turing Machine which does the following:

$$0\underbrace{11\ \ldots\ 1}_{r}\underbrace{\overset{\downarrow}{1}1\ \ldots\ 11}_{n-r}0\underbrace{11\ \ldots\ 1}_{r} \;\mapsto\; 0\underbrace{11\ \ldots\ 1}_{r+1}\underbrace{\overset{\downarrow}{1}1\ \ldots\ 11}_{n-r-1}0\underbrace{11\ \ldots\ 1}_{r+1}$$

Here is the table describing the machine:

| | 0 | 1 |
|---|---|---|
| 0 | | 0R1 |
| 1 | 0R2 | 1R1 |
| 2 | 1L3 | 1R2 |
| 3 | 0L4 | 1L3 |
| 4 | 1R0 | 1L4 |

We check the sequence of actions when we start with $01\overset{\downarrow}{1}11010$:

$$01\overset{\overset{0}{\downarrow}}{1}11010 \xrightarrow{0R1} 010\overset{\overset{1}{\downarrow}}{1}1010 \xrightarrow{1R1} 0101\overset{\overset{1}{\downarrow}}{1}010 \xrightarrow{1R1} 01011\overset{\overset{1}{\downarrow}}{0}10 \xrightarrow{0R2} 010110\overset{\overset{2}{\downarrow}}{1}0 \xrightarrow{1R2} 0101101\overset{\overset{2}{\downarrow}}{0} \xrightarrow{1L3} 010110\overset{\overset{3}{\downarrow}}{1}1 \xrightarrow{1L3}$$

$$01011\overset{\overset{3}{\downarrow}}{0}11 \xrightarrow{0L4} 0101\overset{\overset{4}{\downarrow}}{1}011 \xrightarrow{1L4} 010\overset{\overset{4}{\downarrow}}{1}1011 \xrightarrow{1L4} 01\overset{\overset{4}{\downarrow}}{0}11011 \xrightarrow{1R0} 011\overset{\overset{0}{\downarrow}}{1}1011$$

We notice that we have a loop, and the machine does not have an assignment at the situation

$$0\underbrace{11\ \ldots\ 11}_{n}\overset{\downarrow}{0}\underbrace{11\ \ldots\ 11}_{n}.$$

**Step 2.** Now we describe the machine which will finish the work:

$$0\underbrace{11\ \ldots\ 11}_{n}\overset{\downarrow}{0}\underbrace{11\ \ldots\ 11}_{n} \;\mapsto\; 0\underbrace{\overset{\downarrow}{1}1\ \ldots\ 11}_{n}0\underbrace{11\ \ldots\ 11}_{n}$$

Here is the table describing this machine:

| | 0 | 1 |
|---|---|---|
| 0 | 0L5 | |
| 5 | 0R6 | 1L5 |

We check the actions if we start with $01111\overset{\downarrow}{0}11110$:

$$01111\overset{\overset{0}{\downarrow}}{0}11110 \xrightarrow{0L5} 0111\overset{\overset{5}{\downarrow}}{1}011110 \xrightarrow{1L5} 011\overset{\overset{5}{\downarrow}}{1}1011110 \xrightarrow{1L5} 01\overset{\overset{5}{\downarrow}}{1}11011110 \xrightarrow{1L5} 0\overset{\overset{5}{\downarrow}}{1}111011110 \xrightarrow{1L5}$$

$$\overset{\overset{5}{\downarrow}}{0}1111011110 \xrightarrow{0R6} 0\overset{\overset{6}{\downarrow}}{1}111011110$$

Now we can combine Steps 1 and 2 to give the following design:

| | 0 | 1 |
|---|---|---|
| 0 | 0L5 | 0R1 |
| 1 | 0R2 | 1R1 |
| 2 | 1L3 | 1R2 |
| 3 | 0L4 | 1L3 |
| 4 | 1R0 | 1L4 |
| 5 | 0R6 | 1L5 |

**Example 5.** Now we would like to build a Turing Machine to compute the function $f(n) = 2n$. We will combine two designs: first, we compute the function $f_1(n) = (n, n)$ which is described in Example 4, and then we use the design of the Turing machine computing the sum $f_2(n, n) = n + n$. Here are two designs of the combination:

| | 0 | 1 |
|---|---|---|
| 0 | 0L5 | 0R1 |
| 1 | 0R2 | 1R1 |
| 2 | 1L3 | 1R2 |
| 3 | 0L4 | 1L3 |
| 4 | 1R0 | 1L4 |
| 5 | 0R6 | 1L5 |
| 6 | | 0R7 |
| 7 | 1L8 | 1R7 |
| 8 | 0R9 | 1L8 |

| | 0 | 1 |
|---|---|---|
| 0 | 0L5 | 0R1 |
| 1 | 0R2 | 1R1 |
| 2 | 1L3 | 1R2 |
| 3 | 0L4 | 1L3 |
| 4 | 1R0 | 1L4 |
| 5 | 0R6 | 1L5 |
| 6 | 1L7 | 1R6 |
| 7 | 0R8 | 1L7 |
| 8 | | 0R9 |

**Exercise 1.** Check the instructions to compute $2 \cdot 4$.

**Exercise 2.** For given integer $q \geq 1$, design a Turing Machine computing the function $f(n) = q \cdot n$, where $n \geq 1$.