Math 232, Winter 2017, Boris Botvinnik

REVIEW PROBLEM FOR THE SECOND MIDTERM

1. An algebraic expression is written in the reverse Polish notations as follows:

$$x7 + 3 \wedge x1 - x * /1x2 \wedge 5 + /+$$

(a) Find a binary tree representing this algebraic expression.

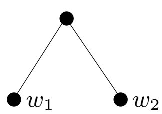
(b) Find this algebraic expression.

(c) Write this expression in the Polish notations.

2. Describe the most effective way how to merge together the ordered lists L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8 with the lengths $|L_1| = 10$, $|L_2| = 15$, $|L_3| = 7$, $|L_4| = 27$, $|L_5| = 37$, $|L_6| = 28$, $|L_7| = 20$, $|L_8| = 9$.

3. Consider the Huffman Algorithm:

Huffman$(L = \{w_1, w_2, \ldots, w_k\})$:
$\{$`Input: A list of weights:` $L = \{w_1, w_2, \ldots, w_k\}$, $k \geq 2\}$
$\{$`Output: an optimal tree` $T(L)\}$
`if` $k = 2$ `then`
`return the tree`



`else`
`Choose two smallest weights` u `and` v `of` L.
`Make a list` L' `by removing the elements` u `and` v `and ading the element` $u + v$.
`Let` $T(L') :=$ **Huffman**(L').
`Form a tree` $T(L)$ `from` $T(L')$ `by replacing a leaf of weight` $u + v$
`by a subtree with two leaves of weights` u `and` v.

`return` $T(L)$.

Prove that the algorithm **Huffman**(L) does produce an optimal binary tree for the weights $L = \{w_1, w_2, \ldots, w_k\}$.

4. Here is the prefix code: $\{00, 10, 11, 011, 111, 1100, 1101, 0100, 0101\}$

(a) Construct a binary tree whose leaves represent this binary code.

(b) Decode the following message:

$$1100110100000111110101110110$$

using the following symbols:

00	10	11	011	111	0100	0101	1100	1101
N	U	H	K	$\smile$	A	Y	T	O

5. Let $\Sigma = \{0, 1\}$ and A_n be the set of binary strings of length n which do not contain the string 00. Find and solve a recurrence relation for $a_n = |A_n|$.

6. Prove that if a finite graph $G = (V, E)$ in which each vertex has degree at least 2 contains a cycle.

7. Prove that if a finite graph $G = (V, E)$ is a tree, then $|V| = |E| + 1$.

8. Let K_n be a complete graph with n vertices. For which n the graph K_n admits an Euler circuit? Explain in detail.

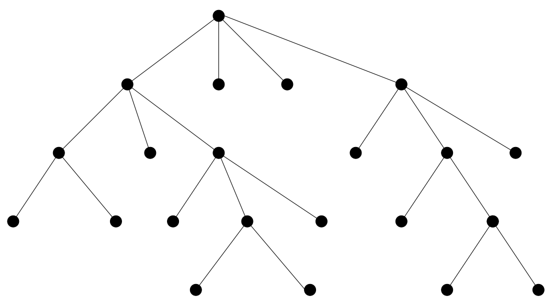
9. Construct an optimal tree for the following weights $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43\}$.

10. Prove that any tree has at least two leaves.

11. Let $T = (V, E)$ be a tree. Prove that for any two distinct vertices $v, u \in V$ there is a unique path connecting them.

12. Let $G = (V, G)$ be a graph with no loops and parallel edges, and $|V| = n \geq 3$. Prove that if $\deg(v) + \deg(w) \geq n$ for each pair of vertices v and w which are not connected by an edge, then G has a Hamiltonian cycle.

13. Give the postorder and preorder listings for the following tree:



14. Let T be a complete binary tree.

(a) Prove that it has odd number of vertices.

(b) Assume the height of T is h, and T has ℓ leaves. Prove that $\ell \leq 2^h$.

15. Write the expression $(x-1)(x^5+x^4+x^3+x^2+x+1)-(x^6-1)$ in Polish notations.

16. Let $G = (V, E)$ be a finite graph.

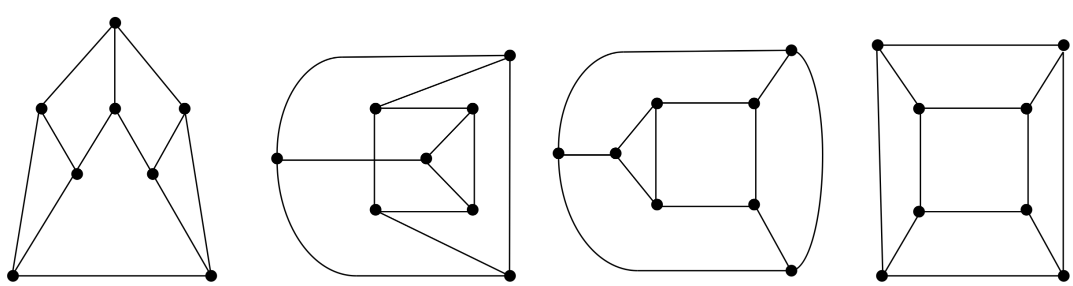
(a) Assume that $|V| = |E| + 1$ and that G is connected. Prove G is a tree.

(b) Assume that $|V| = |E| + 1$. Find an example that G is not a tree.

17. A connected graph $G = (V, E)$ has 50 edges. What is the maximal value of $|V|$? Give proof and example.

18. Let $G = (V, E)$ be a loop-free connected graph with $V = \{v_1, \ldots, v_n\}$, where $n \geq 2$, $\deg v_1 = 1$ and $\deg v_j \geq 2$ for all $2 \leq j \leq n$. Prove that G must have a cycle.

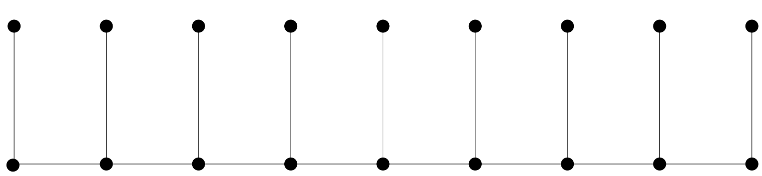
19. Which, if any, of the pairs of graphs shown are isomorphic? Justify your answer by describing an isomorphism or explaining why one does not exist.



G_1 G_2 G_3 G_4

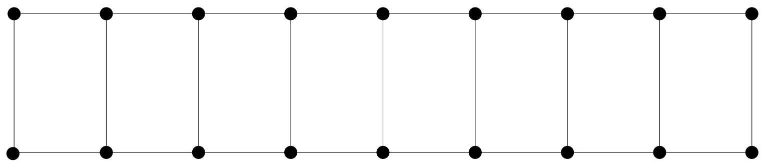
21. Compute the chromatic polynomial of the graphs G_1 and G_4.

22. Compute chromatic polynomial of the following graph G



Find $\chi(G)$.

23. Compute chromatic polynomial of the following graph G



Find $\chi(G)$.

24. Let C_n be a cycle on n vertices. Prove that $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.

25. Let W_{n+1} be a wheel on $(n+1)$ vertices. Compute the chromatic polynomial of W_{n+1}. Find $\chi(W_{n+1})$.