Math 232, Winter 2017, Boris Botvinnik

REVIEW FOR THE FINAL TEST II:

1. Design a recursive algorithm which for each positive integer n computes such k that $7^{k-1} \leq n < 7^k$.

2. Let $\Sigma = \{a, b\}$, and Σ^* be the language over Σ. Describe recursively the set T of words containing at least one a's, at least one b's and where all a's precede all b's.

3. Consider the following algorithm:

CON-3 $[n]$:
{`Input: a non-negative integer` n}
{`Output: ????`} $\longleftarrow$ explain
`if` $n < 3$ `then`
  `return` n
`else`
  `return` **CON-3** $[n$ DIV $3]n$ MOD 3

{Here n MOD 3 follows the number **CON-3** $[n$ DIV $3]$.}

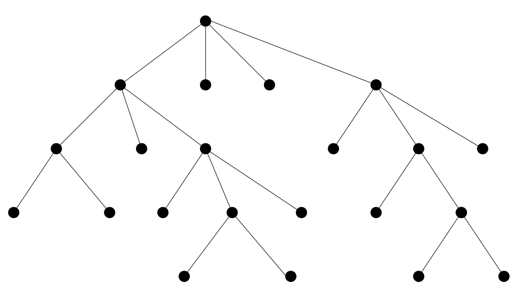
Does the algorithm **CON-3** $[n]$ terminate?

What is the output if you evaluate **CON-3** $[101]$? **CON-3** $[9{,}999]$?

4. Let K_n be a complete graph with n vertices. For which n the graph K_n admits an Euler circuit? Explain in detail.

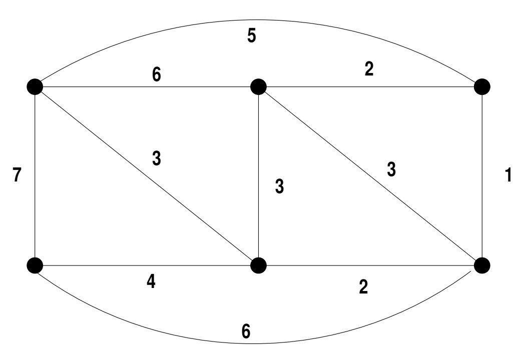
5. Construct an optimal tree for the following weights $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43\}$.

6. Give the postorder and preorder listings for the following tree:



7. Define a complete binary tree. How many complete binary trees are there with $2n + 1$ vertices are there?

8. Write the Prim's algorithm. Use Prim's algorithm to find a minimal spanning tree for the following graph:

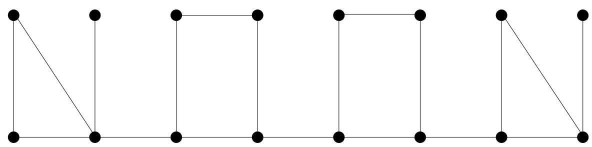


9. Let $G = (V, G)$ be a graph with no loops and parallel edges, and $|V| = n \geq 3$. Give a detailed proof that if $\deg(v) + \deg(w) \geq n$ for each pair of vertices v and w which are not connected by an edge, then G has a Hamiltonian cycle.

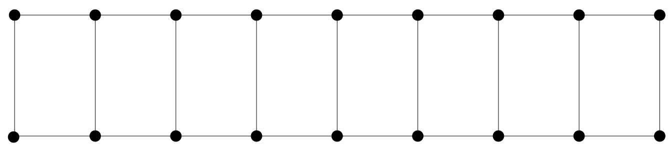
10. Write the Kruskal's algorithm. Use Kruskal's algorithm to find a minimal spanning tree for the weighted graph in problem # 8.

11. Find chromatic polynomials $P(G, \lambda)$ for the following graphs:

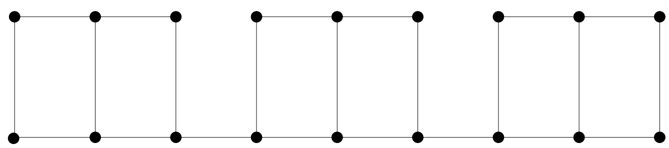
(a)



(b)



(c)



12. Prove that any tree has at least two leaves.

13. Let $T = (V, E)$ be a tree. Prove that for any two distinct vertices $v, u \in V$ there is a unique path connecting them.

14. Let $T = (V, E)$ be a tree. Prove that $|V| = |E| + 1$.

15. Consider the Prim's Algorithm:

Prim(G):=**Prim**$(G = (v, E),\ \{w_1 = w(e_1), \dots, w_k = w(e_k),\ k = |E|\},\ G$ is a connected graph$)$
$\{$`Output: a minimal spanning tree` $T = (V_T, E_T)\}$:
`set` $V_T := \emptyset$, $E_T := \emptyset$.
    `Choose a vertex` $w \in V$, `set` $V_T := \{w\}$.
        `While` $|V_T| < |V|$ `do`
            `Choose an edge` $e = \{u, v\}$ `of minimal weight with` $u \in V_T$, $v \in V \setminus V_T$.
            `Set` $V_T := V_T \cup \{v\}$, $E_T := E_T \cup \{e\}$.
        `Return` $T := (V_T, E_T)$.

Prove that the algorithm **Prim**(G) does produce a minimal spanning tree.

16. Let T be a complete binary tree.

- **(a)** Prove that it has odd number of vertices.
- **(b)** Assume the height of T is h, and T has ℓ leaves. Prove that $\ell \leq 2^h$.

17. Write the expression $(x - 1)(x^5 + x^4 + x^3 + x^2 + x + 1) - (x^6 - 1)$ in Polish notations.

18. Let $G = (V, E)$ be a finite graph.

- **(a)** Assume that $|V| = |E| + 1$ and that G is connected. Prove G is a tree.
- **(b)** Assume that $|V| = |E| + 1$. Find an example that G is not a tree.

19. A connected graph $G = (V, E)$ has 50 edges. What is the maximal value of $|V|$? Give proof and example.

20. Let $G = (V, E)$ be a loop-free connected graph with $V = \{v_1, \dots, v_n\}$, where $n \geq 2$, $\deg v_1 = 1$ and $\deg v_j \geq 2$ for all $2 \leq j \leq n$. Prove that G must have a cycle.