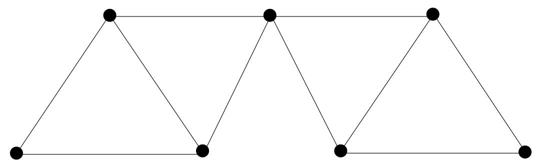
Math 232, Winter 2017, Boris Botvinnik

REVIEW FOR THE FINAL TEST, PART I:

1. Determine the chromatic polynomial $P(G, \lambda)$ for the following graph. Find a number of proper colorings of the graph G with 5 colors.



2. An algebraic expression is written in the reverse Polish notations as follows:

$$- \; / \; - \; \wedge \; x \; 3 \; 1 \; - \; x \; 1 \; + \; \wedge \; x \; 2 \; + \; x \; 1$$

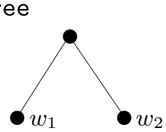
(a) Find a binary tree representing this algebraic expression.

(b) Find this algebraic expression.

(c) Write this expression in the Polish notations.

3. Consider the Huffman Algorithm:

Huffman$(L = \{w_1, w_2, \ldots, w_k\})$:
{Input: A list of weights: $L = \{w_1, w_2, \ldots, w_k\}$, $k \geq 2$}
{Output: an optimal tree $T(L)$}
if $k = 2$ then
return the tree



else
Choose two smallest weights u and v of L.
Make a list L' by removing the elements u and v and ading the element $u + v$.
Let $T(L') :=$**Huffman**(L').
Form a tree $T(L)$ from $T(L')$ by replacing a leaf of weight $u + v$
by a subtree with two leaves of weights u and v.

return $T(L)$.

Prove that the algorithm **Huffman**(L) does produce an optimal binary tree for the weights $L = \{w_1, w_2, \ldots, w_k\}$.

4. Find an Euler circuit (if it does exist) in a given graph.

5. Describe the most effective way how to merge together the ordered lists L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9 with the lengths $|L_1| = 17$, $|L_2| = 12$, $|L_3| = 9$, $|L_4| = 30$, $|L_5| = 32$, $|L_6| = 44$, $|L_7| = 15$, $|L_8| = 10$, $|L_9| = 44$.

6. Use generating functions to resolve the recurrence relation: $x_0 = 2$, $x_n = 3x_{n-1} - 4n$.

7. Let $\Sigma = \{0, 1, 2\}$ and A_n be the set of binary strings of length n which do not contain the string 01. Find and solve a recurrence relation for $a_n = |A_n|$.

8. Solve the following recurrence relations:

(a) $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$,
$a_0 = 0$, $a_1 = 1$.

(b) $a_n = 2a_{n-1} - 2a_{n-2}$, $n \geq 2$,
$a_0 = 1$, $a_1 = 1$.

9. Prove that if a finite graph $G = (V, E)$ in which each vertex has degree at least 2 contains a cycle.

10. Prove that if a finite graph $G = (V, E)$ is a tree, if and only if $|V| = |E| + 1$.

11. Let G_n be a graph which is obtained from the complete graph K_n by deleting one edge. Determine the chromatic polynomial $P(G_n, \lambda)$ and the chromatic number $\chi(G_n)$.

12. Use generating functions to solve the following recurrence relations

(a) $a_n = a_{n-1} + n$ for $n \geq 1$, and $a_0 = 1$;

(b) $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 1$, $a_1 = -2$;

(c) $a_n = -3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n$, $n \geq 2$, and $a_0 = 2$, $a_1 = 1$.

13. Find and solve a recurrence relation for a_n, the number of 012-strings of length n in which no 2 is followed (immediately or after some intervening characters) by a 1.

14. Let a_n be the number of words of lenght n in A, B, C, and D with an odd number of B's. Calculate a_0, a_1, a_2, a_3, a_4. Find a recurrence relation satisfied by a_n for all $n \geq 2$.

15. Consider the Euclid Algorithm:

```
EUCLID(m, n):
{Input:  m, n ∈ N, not both 0}
{Output:  gcd(m, n)}
if n = 0 then
   return m
else
   return EUCLID(n, m MOD n)
```

Use the algorithm to find gcd(21,231, 141).

16. Let $G = (V, E)$ be a loop-free finite graph with $|V| = n$. Prove that G is a tree if and only if its chromatic polynomial $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$

17. Let C_n be a cycle of length n. Prove that $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.

18. Let W_{n+1} be a "wheel" with $n + 1$ vertices. Prove that $P(W_{n+1}, \lambda) = \lambda(\lambda - 2)^n + (-1)^n\lambda(\lambda - 2)$.

19. Here is the algorithm **Tree**.

```
Tree(G, v)
Input: A vertex v of the finite graph G
Output: A set E of edges of a spanning tree for the component of G that contains v
Let V := {v} and E := ∅
   (where V is a list of visited vertices).
while there are edges of G joining vertices in V to vertices that are not in V do
   Choose such an edge {u, w} with u ∈ V and w ∉ V.
   Put w in V and put the edge {u, w} in E.
return E
```

Prove that **Tree**(G, v) produces a spanning tree for the component of G containing the vertex v.

20. Here is the Kruskal's Algorithm:

```
Kruskal's Algorithm(G = (V(G), E(G)), wt : E(G) → (0, ∞))
Input: A finite weighted connected graph (G, wt) with edges listed in order of increasing weight
Output: A set E of edges of an optimal spanning tree for G)
Set E = ∅, for j = 1 to |E(G)| do
   if E ∪ {e_j} is acyclic then
   Put e_j in E.
return E
```

Prove that **Tree**(G, v) produces a spanning tree for the component of G containing the vertex v.