Summary on Lecture 4, January 18, 2017

Second Order Recurrence Relations (continuation)

Example: legal arithmetic expressions without parenthesis. In most computing languages, it important to use "legal arithmetic expressions without parenthesis". These expressions are made up out of the digits 0,1,..., 9 and binary symbols $+, *, /$. For example, the expressions $7+8$, $5+7*3$, $33*7+4+6*4$ are legal expressions, and the expressions $/7+8$, $5+7*3+$, $33*7+/4+6*4$ are not.

We denote by a_n the number of legal expressions of length n. Then $a_1 = 10$ since the only legal expressions of length 1 are the digits $0, 1, \ldots, 9$. Then $a_2 = 100$ which accounts for the expressions 00, 01, ..., 99.

Let $n \geq 3$. We observe:

(1) Let x be an arithmetic legal expression of $(n-1)$ symbols. Then the last symbol must be a digit. We add one more digit to the right of x and obtain $10x$ more legal expressions of the length n.

(2) Let y be an arithmetic legal expression of $(n-2)$ symbols. Then we can add to the right of y one of the following 29 2-symbol expressions: $+0$, $+1$, ..., $+9$, $*0$, $*1$, ..., $*9$, $/1$, ..., $/9$ (no division by 0 is allowed).

We obtain the recurrence relation: $a_0 = 10$, $a_1 = 10$, $a_n = 10a_{n-1} + 29a_{n-2}$ for $n \geq 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 10$, $a_1 = 10$, $a_n = 10a_{n-1} + 29a_{n-2}$, $n \geq 3$.

Example. We would like to find a number of binary sequences of the length n without any consecutive 0's.
Let a_n denote the number of such sequences of length $n \geq 1$. Clearly, if $n = 1$, we have 0, 1, i.e., $a_1 = 2$, if $n = 2$, we have the sequences 01, 10, 11, i.e., $a_2 = 3$.
Let $n \geq 3$. Let $x_1 \cdots x_{n-2}x_{n-1}x_n$ be a sequence like that. There are two cases:

(1) The last symbol $x_n = 1$. Then the sequence $x_1 \cdots x_{n-2}x_{n-1}$ has no consecutive 0's.

(2) The last symbol $x_n = 0$. Then $x_{n-1} = 1$, and the sequence $x_1 \cdots x_{n-2}$ has no consecutive 0's.

Thus we conclude that $a_n = a_{n-1} + a_{n-2}$. Also we notice that the initial conditions $a_1 = 2$, $a_2 = 3$ could be replaced by $a_0 = 1$, $a_1 = 2$. Then $a_2 = a_1 + a_0 = 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

The case of complex roots. Let $z = x + iy \in \mathbf{C}$ be a complex number. Then we let $|z| = \sqrt{x^2 + y^2}$, and we can write z as

$$z = |z|(\cos\theta + i\sin\theta), \quad \cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}.$$

There is a standard notation $e^{i\theta} := \cos\theta + i\sin\theta$. There is a remarkable formula (DeMoivre Theorem):

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{or} \quad (e^{i\theta})^n = e^{in\theta}$$

We prove it by induction. Clearly this formula holds for $n = 1$. Assume it holds for $n = k$. Then we have:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta) \\
&= \cos(k+1)\theta + i\sin(k+1)\theta.
\end{aligned}$$

Here we used the formulas:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

Example. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} - 2a_{n-2}$. Then again, we are looking for a solution as $a_n = cr^n$, $c \neq 0$. We have substitute $a_n = cr^n$ to our recurrence relation:

$$cr^n = 2cr^{n-1} - 2cr^{n-2} \quad \text{or} \quad r^2 - 2r + 2 = 0.$$

We find the solutions of the characteristic equation:

$$r_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm i.$$

Then we have:

$$\begin{aligned} r_1 &= 1+i &= \sqrt{2}(\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}) &= \sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}), \\ r_2 &= 1-i &= \sqrt{2}(\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}) &= \sqrt{2}(\cos\tfrac{\pi}{4} - i\sin\tfrac{\pi}{4}). \end{aligned}$$

Now we are looking for a solution in the form $a_n = c_1 r_1^n + c_2 r_2^n$. We notice the following:

$$\begin{aligned} a_n &= c_1(1+i)^n + c_2(1-i)^n \\ &= c_1\left(\sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4})\right)^n + c_2\left(\sqrt{2}(\cos\tfrac{\pi}{4} - i\sin\tfrac{\pi}{4})\right)^n \\ &= c_1\left(\sqrt{2}\right)^n(\cos\tfrac{n\pi}{4} + i\sin\tfrac{n\pi}{4}) + c_2\left(\sqrt{2}\right)^n(\cos\tfrac{n\pi}{4} - i\sin\tfrac{n\pi}{4}) \\ &= \left(\sqrt{2}\right)^n(K_1\cos\tfrac{n\pi}{4} + K_2\sin\tfrac{n\pi}{4}), \end{aligned}$$

where $K_1 = c_1 + c_2$, $K_2 = i(c_1 - c_2)$. Clearly we would like to find **real values** of K_1 and K_2. We substitute $n = 0$ and $n = 1$ to get the system:

$$\begin{cases} K_1\cos 0 + K_2\sin 0 &= a_0 = 1 \\ \sqrt{2}(K_1\cos\frac{\pi}{4} + K_2\sin\frac{\pi}{4}) &= a_1 = 2 \end{cases} \quad \text{or} \quad \begin{cases} K_1 &= 1 \\ K_1 + K_2 &= 2 \end{cases} \quad \text{or} \quad \begin{cases} K_1 &= 1 \\ K_2 &= 1 \end{cases}$$

We obtain the answer:

$$a_n = \left(\sqrt{2}\right)^n(\cos\tfrac{n\pi}{4} + \sin\tfrac{n\pi}{4})$$

General Case. Now we assume that we have a second order recurrence relation, i.e. a_0, a_1, are given and $a_n = Aa_{n-1} + Ba_{n-2}$, $n \geq 2$, where the coefficients A and B are real numbers (in fact, they are integers in all our examples). Then the characteristic equation is given as $r^2 - Ar - B = 0$. Assume that the roots r_1, r_2 are complex. Obviously, it means that r_1 and r_2 are conjugate, i.e., we can write $r_1 = \rho(\cos\theta + i\sin\theta)$, and $r_2 = \rho(\cos\theta - i\sin\theta)$. Thus we may look for a solution for a_n in the form:

$$\begin{aligned} a_n &= c_1 r_1^n + c_2 r_2^n \\ &= c_1\rho^n(\cos n\theta + i\sin n\theta) + c_2\rho^n(\cos n\theta - i\sin n\theta) \\ &= \rho^n((c_1+c_2)\cos n\theta + i(c_1-c_2)\sin n\theta) \\ &= \rho^n(K_1\cos n\theta + K_2\sin n\theta). \end{aligned}$$

Here $K_1 = (c_1+c_2)$ and $K_2 = i(c_1-c_2)$. In particular, it means that the expression $a_n = \rho^n(K_1\cos n\theta + K_2\sin n\theta)$ satisfies the recurrence relation we started with. We also notice that K_1 and K_2 are assumed to be real. Since a_0 and a_1 are given, we find them by substituting $n = 0$ and $n = 1$:

$$\begin{cases} K_1\cos 0 + K_2\sin 0 &= a_0 \\ \rho(K_1\cos\theta + K_2\sin\theta) &= a_1 \end{cases} \quad \text{or} \quad \begin{cases} K_1 &= a_0 \\ \rho(K_1\cos\theta + K_2\sin\theta) &= a_1 \end{cases}$$

We notice that the system always has a solution provided $\rho \sin\theta \neq 0$ for arbitrary initial coefficients a_0 and a_1. On the other hand, the condition $\rho \sin\theta = 0$ means that either $\rho = 0$ or $\sin\theta = 0$. Each of those imply that the roots r_1 r_2 are real.

We summarize the above discussion:

Theorem 3. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero real constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two complex roots

$$r_1 = \rho(\cos\theta + i\sin\theta), \quad r_2 = \rho(\cos\theta - i\sin\theta),$$

Then $a_n = \rho^n(K_1 \cos n\theta + K_2 \sin n\theta)$, where the coefficients K_1, K_2 are determined by solving the system

$$\begin{cases} K_1 & = & a_0 \\ \rho(K_1 \cos\theta + K_2 \sin\theta) & = & a_1 \end{cases}$$