Summary on Lecture 13, February 13, 2017

Trees: definitions and basic properties

Definition. A connected graph $G = (V, E)$ is a *tree* if G has no cycles.

Theorem 1. *Let $T = (V, E)$ be a tree, and $v, v' \in V(T)$. Then there exists a unique path from v to v'.*

Proof. Assume there are two different paths connecting v and v'. Then there is a cycle. □

Definition. Let $G = (V, E)$ be a graph. A tree $T \subset G$ is a *spanning tree of G* if $V(T) = V(E)$.

Theorem 2. *Let $G = (V, E)$ be a finite graph. Then G is connected if and only if there exists a spanning tree T of G.*

Proof. Let $T \subset G$ be a spanning tree. Then for any two vertices $v, v' \in V(T) = V(G)$ there exists a path in T (and, consequently, in G) which connects v and v'.

On the other hand, if G is connected, then we remove all loops from G to obtain graph G_1, which is still connected. If G_1 is a tree, we are done. If not, we find a cycle in G_1, and choose an edge e_1 in that cycle. Then $G_2 = G_1 \setminus \{e_1\}$ is a connected graph. If G_2 has a cycle, we repeat the procedure. Finally we will obtain a connected graph $G_k \subset G$ which has no cycles since the original graph is finite, i.e., G_k is a spanning tree. □

Let $G = (V, E)$ be a graph. We say that a vertex $v \in V$ is a *leaf* of G if $\deg v = 1$. We need the following observation:

Lemma 1. *Let $T = (V, E)$ be a finite tree. There there are at least two leaves v, v' in T.*

Proof. Consider a longest path in T, and let v and v' are its end-points. We notice that $v \neq v'$ since T is a tree. Then v and v' are both leaves, indeed, if not, we can extend a path by at least one edge. □

Theorem 2. *Let $T = (V, E)$ be a finite tree. Then $|V| = |E| + 1$.*

Proof. Induction on $k = |V|$. Theorem 2 obviously holds if $k = 1$ and $k = 2$. Assume Theorem 2 holds for all trees $T' = (V', E')$ with $|V| \leq n$. Consider a tree $T = (V, E)$ with $|V| = n + 1$. Then by Lemma 1, there exists a leaf $v \in V'$ with a single edge e. We prune the tree T at v, to obtain a tree $T' = (V', E')$, where $V' = V \setminus \{v\}$, $E' = E \setminus \{e\}$. By induction, $|V'| = |E'| + 1$. Since $|V| = |V'| + 1$ and $|E| = |E'| + 1$, we obtain that $|V| = |E| + 1$. □

Theorem 3. *The following statements are equivalent:*

(a) *A graph $G = (V, E)$ is a finite tree.*

(b) *A graph $G = (V, E)$ is connected, but a removal of any edge will make it disconnected.*

(c) *A graph $G = (V, E)$ contains no cycles and $|V| = |E| + 1$.*

(d) *A graph $G = (V, E)$ is connected and $|V| = |E| + 1$.*

Proof. **(a)$\Longrightarrow$(b)** Let $G = (V, E)$ be a finite tree. Assume that a removing an edge e from G keeps $G \setminus \{e\}$ connected. Let v, v' be the end-vertices of e. Then there exists a path in $G \setminus \{e\}$ connecting v and v'. Then we put the edge e back an we obtain a cycle. Thus G could not be a tree in the first place. Contradiction. Thus (a)$\Longrightarrow$(b).

(b)$\Longrightarrow$(c) Let $G = (V, E)$ be as in (b), however, G contains a cycle. Then we can remove an edge e from such a cycle, and the graph $G \setminus \{e\}$ is still connected. Contradiction. Thus $G = (V, E)$ has no cycles, and, by definition, G is a tree. By Theorem 2, $|V| = |E| + 1$.

(c)$\Longrightarrow$(d) Let $G = (V, E)$ be as in (c), however, G is not connected. It means that $G = G_1 \cup \cdots \cup G_r$, where G_i are connective components of G and $r \geq 2$. Then every component G_i has no cycles and it is connected,

thus G_i is a tree by definition. Theorem 2 gives that

$$|V(G_1)| = |E(G_1)| + 1, \ \cdots \ |V(G_r)| = |E(G_r)| + 1.$$

We obtain:
$$|V(G)| = |V(G_1)| + \cdots + |V(G_r)| = |E(G_1)| + \cdots + |E(G_r)| + r = |E(G)| + r.$$

At the same time, we have that $|V(G)| = |E(G)| + 1$. Thus $r = 1$. Contradiction.

(d) $\Longrightarrow$ **(a)** Let $G = (V, E)$ be as in (d). Then G has a spanning tree $T \subset G$ by Theorem 1. Then $V(T) = V(G)$ and $|E(T)| \leq |E(G)|$. Since T is a tree, $|V(T)| = |E(T)| + 1$, and by assumption, $|V(G)| = |E(G)| + 1$. Thus $|E(T)| = |E(G)|$, i.e., $T = G$. In particular, G is a tree (in fact, it is its own spanning tree). $\qquad\square$