

Summary on Lecture 11, February 8, 2017

Graph Coloring and Chromatic Polynomials

Graph coloring, or more specifically vertex coloring means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

This definition allows us to use a separate color for each vertex. From a mathematical perspective graph coloring is only interesting if we restrict the permissible colors to a fixed finite set S . It is easy to see that the choice of actual colors is irrelevant, and therefore any graph property related to coloring may only depend on the cardinality $|S| = k$. We may as well label the nodes using the numbers $1, 2, \dots, k$.

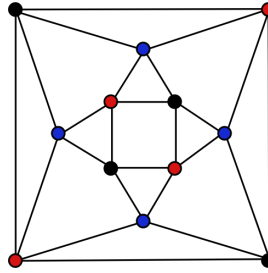


Fig. 1. Graph G_1 with $\chi(G_1) = 3$

Definition. Let $G = (V, E)$ be a graph. A *proper λ -coloring* of a graph G is a function $\sigma : V \rightarrow \{1, 2, \dots, \lambda\}$ which satisfies $\sigma(v) \neq \sigma(v')$ for any edge $e = \{v, v'\}$. Note that it is not compulsory to use all the colors. The graph is said to be *λ -colorable* if such a function exists. The *chromatic number* $\chi(G)$ is the minimal λ for which the graph G is λ -colorable.

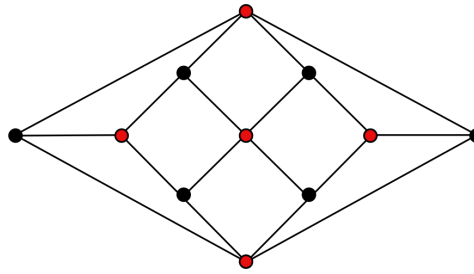


Fig. 2. Graph G_2 with $\chi(G_2) = 2$

Example. It is easy to see that $\chi(K_n) = n$.

In order to determine the chromatic number $\chi(G)$, we consider more general problem of finding the chromatic polynomial $P(\lambda, G)$.

Definition. For each $\lambda = 0, 1, 2, \dots$ the value of the chromatic polynomial $P(G, \lambda)$ is the number of different proper λ -colorings of G . Here we identify colorings $\sigma : V \rightarrow \{1, 2, \dots, \lambda\}$ and $\sigma' : V \rightarrow \{1, 2, \dots, \lambda\}$ if $\sigma = \sigma'$ as functions.

Examples. (1) Let $G = (V, E)$ with $|V| = n$ and $E = \emptyset$. Then $P(G, \lambda) = \lambda^n$.

(2) $P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$. We use the notation $\lambda^{(n)} := \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$.

(3) Let C be a path with n vertices. Then $P(C, \lambda) = \lambda(\lambda - 1)^{n-1}$. Similarly, $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$ for any tree with n vertices.

Let $G = (V, E)$ be a graph, and e be its edge with vertices a and b . We denote by G_e the graph which is obtained by removing the edge e . Let G'_e be a graph which is obtained from G_e by identifying the vertices a and b , see Fig. 3.

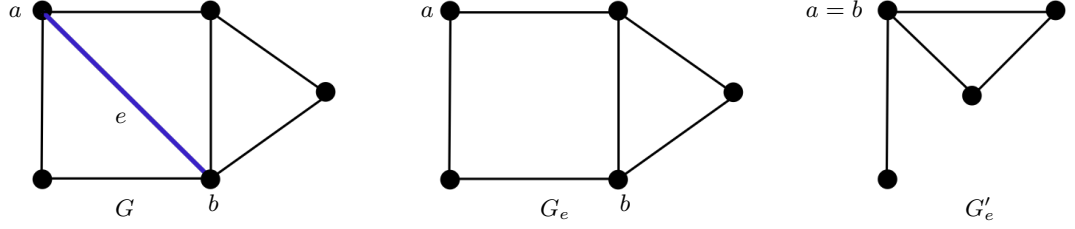


Fig. 3. The graphs G , G_e and G'_e

Theorem 1. Let $G = (V, E)$ be a connected graph, and $e \in E$. Then

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda).$$

Proof. Let $e = \{a, b\}$. Consider the value $P(G_e, \lambda)$. There are two possibilities here: either the vertices a and b have the same color or not. If they are of different colors, then it corresponds to a proper coloring of G . If they are the same, then it corresponds to a proper coloring of G'_e . \square

Lemma 1. Let T be a tree with n vertices. Then $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof. Induction on n . If $n = 1$, then obviously $P(T, \lambda) = \lambda$. Let $n > 1$. We find an edge e such that $e = \{a, b\}$, where a is a leaf. Then T_e is a disjoint union of a tree on $(n - 1)$ vertices and a single vertex, and T'_e is a tree on $(n - 1)$ vertices, see Fig. 4

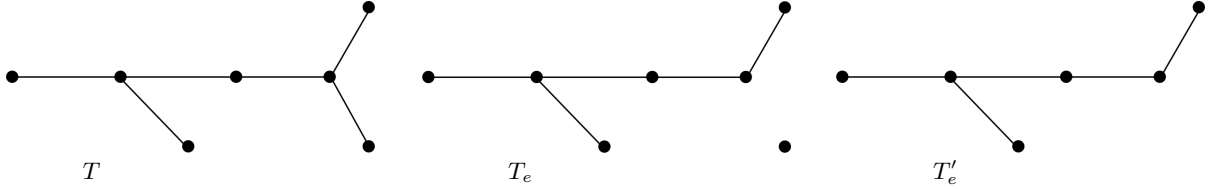


Fig. 4. The graphs T , T_e and T'_e

By induction, we have that $P(\lambda, T'_e) = \lambda(\lambda - 1)^{n-2}$, and $P(\lambda, T_e) = \lambda(\lambda - 1)^{n-2} \cdot \lambda = \lambda^2(\lambda - 1)^{n-2}$. Then

$$P(\lambda, T) = P(\lambda, T_e) - P(\lambda, T'_e) = \lambda^2(\lambda - 1)^{n-2} - \lambda(\lambda - 1)^{n-2} = \lambda(\lambda - 1)^{n-1}.$$

This completes the induction. \square