Summary on Lecture 7, January 15, 2016

Introduction to Graph Theory. Definitions and Examples.

A graph G is given by three objects: a set $V = V(G)$ of vertices, a set $E = E(G)$ of edges and an assignment of the end vertices to every edge.

We will distinguish graphs and directed graphs (or digraphs), where each edge has a direction and its two vertices could be thought as its begining and its end.

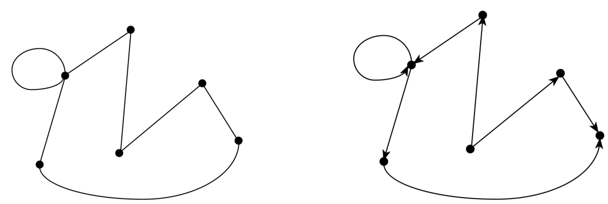

Fig. 1. Graph and digraph

Definition 1. Let V be an non-empty set, and $E \subset V \times V$. Then the pair $G = (V, E)$ is a *directed graph*. If

$$E \subset \{\{v_0, v_1\} \mid v_0, v_1 \in V\ \}$$

(where $\{\{v_0, v_1\} \mid v_0, v_1 \in V\ \}$ is the set of two-element subsets of V), then the pair $G = (V, E)$ is a *graph*, see Fig. 1. The set $V = V(G)$ is a set of vertices, and $E = E(G)$ is a set of edges of G.

It is convenient to denote an edge e as a pair of verices $e = \{v, v'\}$ (for a graph) and as an ordered pair $e = (v, v')$ (for a digraph). A loop is an edge e with the same vertices $v = v'$. Below we assume that a graph (or digraph) G has no loops.

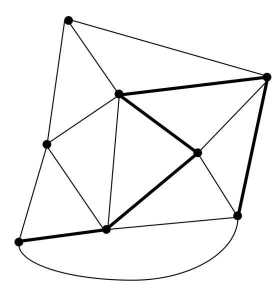

Fig. 2. A walk in G

Let $G = (V, E)$ be a graph, and $x, x' \in V$ be two vertices. An x-x'-*walk* is a finite alternatig sequence

$$x = x_0, e_1, x_1, e_2, x_2, \ldots, x_{n-1}, e_n, x_n = x'$$

of vertices and edges, where $e_i = \{x_{i-1}, x_i\}$ for $i = 1, 2, \ldots, n$, see Fig. 2. If $x = x'$, then an x-x'-walk is a *clossed walk*. Otherwise, the walk is *open*. There are special types of walks:

- If no edge is repeated, then an x-x'-walk is an x-x'-*trail*.
- A closed trail is called a *circuit*.
- If no vertex is repeated, the an x-x'-walk is called an x-x'-*path*.
- If $x = x'$, then an x-x'-path is called a *cycle*.

Theorem 1. *Let $G = (V, E)$ be a graph with $x, x' \in V$, $x \neq x'$. If there exists an x-x'-trail, then there exists an x-x'-path.*

Proof. Since there exists a trail from x to x', then there exists a trail of a shortest length. Let

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$$

be such a trail. If it is not a path, then $x_j = x_i$ for some $j < i$, i.e., the shortest trail is given as

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{j-1}, x_j\}, \{x_j, x_{j+1}\}, \ldots, \{x_i, x_{i+1}\}, \ldots, \{x_{n-1}, x_n\}.$$

Here $x_j = x_i$, thus we can make an x-x'-trail shorter:

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{j-1}, x_j\}, \{x_i, x_{i+1}\}, \ldots, \{x_{n-1}, x_n\}.$$

This completes the proof. □

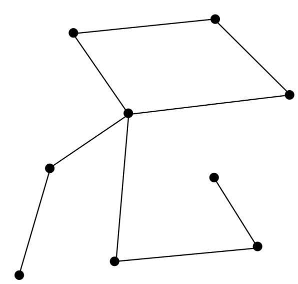

Fig. 3. Finding a shortest trail

A graph G is *path-connected* if any two vertices are connected by a trail (and, consequently, by a path). Also we say that vertices $x, x' \in V$ are in the same path component of G if there exists an x-x'-trail. This relation on the set of vertices is an equivalence relation. Indeed:

- There is always a *trivial x-x-trail*, i.e., $x \sim x$ (reflexivity).
- If there is an x-x'-trail, then there is x'-x-trail, i.e., if $x \sim x'$, then $x' \sim x$ (symmetry).
- If there is an x-x'-trail and there is an x'-x''-trail , then there is x-x''-trail, i.e., if $x \sim x'$ and $x' \sim x''$, then $x \sim x''$ (transitivity).

This equivalence relation breakes a graph G into path-components: $G = G_1 \sqcup \cdots G_s$, such that each graph G_i is path-connected.

We say that two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there are bijections $\phi : V \to V'$ and $\Phi : E \to E'$, such that for each edge $e = \{v, v'\} \in E$,

$$\Phi(e) = \Phi(\{v, v'\}) = \{\phi(v), \phi(v')\}.$$

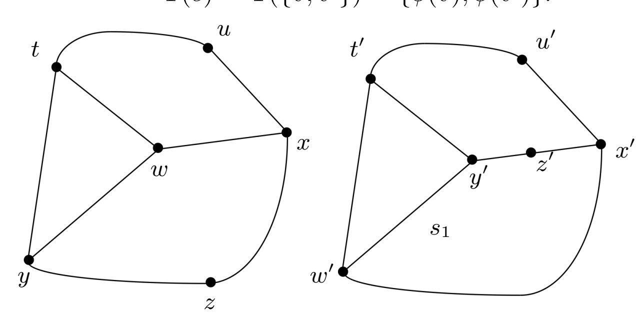

Fig. 4. Two isomorphic graphs

It is often useful to count the number of edges attached to a particular vertex. To get the right count, we need to treat loops differently from edges with two distinct vertices. We define $\deg(v)$, the degree of the vertex $v \in V(G)$, to be the number of 2-vertex edges with v as a vertex plus twice the number of loops with v as vertex. If you think of a picture of G as being like a road map, then the degree of v is simply the number of roads you can take to leave v, with each loop counting as two roads.

The number $D_k(G)$ of vertices of degree k in G is an isomorphism invariant, as is the degree sequence $(D_0(G), D_1(G), D_2(G), \ldots)$.

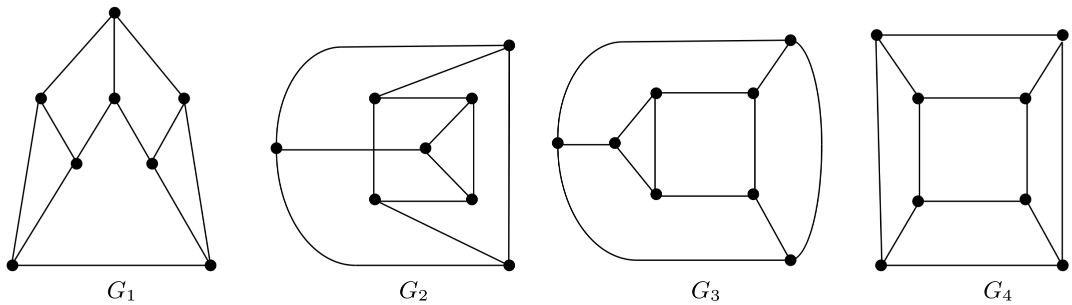

Fig. 5. Isomorphic and non-isomorphic graphs

Exercise 1. Find particular isomorphisms for the graphs G_1, G_2 and G_3 from Fig. 5. Show that the graphs G_1, G_2 and G_3 are not isomorphic to the graph G_4.

Remark. Notice that the graphs G_1, G_2, G_3 and G_4 have the same number of vertices of the same degree. Thus having the same degree sequences does not guarantee that graphs are isomorphic.