Summary on Lecture 6, January 13, 2016

The Method of Generating Functions: More examples.

Example 2. Let $a_0 = 3$, $a_1 = 7$ and $a_{n+2} - 5a_{n+1} + 6a_n = 2$, $n \geq 2$. This is also a *non-homogeneous* new type of recurrence relation. We multiply the relation by x^{n+2}:

$$a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}$$

and we obtain the identity

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}. \tag{1}$$

We denote $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}x^{n+2} &= f(x) - a_0 - a_1 x &&= f(x) - 3 - 7x \\
\sum_{n=0}^{\infty} a_{n+1}x^{n+2} &= x(f(x) - a_0) &&= x(f(x) - 3) \\
\sum_{n=0}^{\infty} a_n x^{n+2} &= x^2 f(x) \\
\sum_{n=0}^{\infty} x^{n+2} &= \frac{x^2}{1-x}
\end{aligned}$$

Then the identity (1) turns to the equation for $f(x)$:

$$\begin{aligned}
(f(x) - 3 - 7x) - 5x(f(x) - 3) + 6x^2 f(x) &= \frac{2x^2}{1-x} \quad \text{or} \\
f(x)(1 - 5x + 6x^2) &= \frac{2x^2}{1-x} + 3 - 8x
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
f(x) &= \frac{(3-8x)(1-x) + 2x^2}{(1-x)(1-5x+6x^2)} \\
&= \frac{3 - 11x + 10x^2}{(1-x)(1-5x+6x^2)} \\
&= \frac{(3-5x)(1-2x)}{(1-x)(1-2x)(1-3x)} = \frac{3-5x}{(1-x)(1-3x)}
\end{aligned}$$

Here we used that $3 - 11x + 10x^2 = (3-5x)(1-2x)$ and $1 - 5x + 6x^2 = (1-2x)(1-3x)$. Now we write

$$\frac{3-5x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x} = \frac{A - 3Ax + B - Bx}{(1-x)(1-3x)} = \frac{(A+B) - (3A+B)x}{(1-x)(1-3x)}$$

which yields the system:

$$\left\{\begin{aligned} A + B &= 3 \\ 3A + B &= 5 \end{aligned}\right. \quad \text{or} \quad \left\{\begin{aligned} A &= 1 \\ B &= 2 \end{aligned}\right.$$

This gives the generating function:

$$f(x) = \frac{1}{1-x} + \frac{2}{1-3x} = \sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}(1+2\cdot 3^n)x^n.$$

We obtain the solution: $a_n = 1 + 2\cdot 3^n$, $n \geq 0$.

Example 3. Let $a_0 = 0$, $a_1 = 6$ and $a_n = -3a_{n-1} + 10a_{n-2} + 3\cdot 2^n$, $n \geq 2$. This is also a *non-homogeneous* new type of recurrence relation. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function. We multiply by x^n the recurrence relation to get

$$\sum_{n=2}^{\infty} a_n x^n = -3\sum_{n=2}^{\infty} a_{n-1}x^n + 10\cdot\sum_{n=2}^{\infty} a_{n-2}x^n + 3\cdot\sum_{n=2}^{\infty} 2^n x^n$$

We notice:

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n x^n &= f(x) - a_1 x - a_0 &&= f(x) - 6x \\
\sum_{n=2}^{\infty} a_{n-1} x^n &= x(f(x) - a_0) &&= xf(x) \\
\sum_{n=2}^{\infty} a_{n-2} x^n &= x^2 f(x) \\
\sum_{n=2}^{\infty} 2^n x^n &= \frac{1}{1-2x} - (1+2x)
\end{aligned}$$

Then we obtain

$$f(x) - 6x = -3xf(x) + 10x^2 + \frac{3}{1-2x} - 3(1+2x)$$

We simplify:

$$f(x)(1+3x-10x^2) = \frac{3}{1-2x} - 3 = \frac{6x}{1-2x}$$

Since $1 + 3x - 10x^2 = (1+5x)(1-2x)$, we obtain

$$f(x) = \frac{6x}{(1-2x)^2(1+5x)}.$$

We use the same method as above:

$$\frac{6x}{(1-2x)^2(1+5x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+5x}$$

We obtain:

$$\begin{aligned}
6x &= A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2 \\
&= (A+B+C) + (3A+5B-4C)x + (-10A+4C)x^2
\end{aligned}$$

We solve the system

$$\begin{cases} A + B + C &= 0 \\ 3A + 5B - 4C &= 6 \\ -10A \quad + 4C &= 0 \end{cases} \Longrightarrow \begin{cases} A &= -\frac{12}{49} \\ B &= \frac{6}{7} \\ C &= -\frac{30}{49} \end{cases}$$

We obtain:

$$\begin{aligned} f(x) &= -\frac{12}{49}\cdot\frac{1}{1-2x}+\frac{6}{7}\cdot\frac{1}{(1-2x)^2}-\frac{30}{49}\cdot\frac{1}{1+5x} \\ &= -\frac{12}{49}\cdot\sum_{n=0}^{\infty}2^n x^n+\frac{6}{7}\cdot\sum_{n=0}^{\infty}(n+1)2^n x^n-\frac{30}{49}\cdot\sum_{n=0}^{\infty}5^n x^n \\ &= \sum_{n=0}^{\infty}\left(-\frac{12}{49}\cdot 2^n+\frac{6}{7}\cdot(n+1)2^n-\frac{30}{49}\cdot 5^n\right)x^n \end{aligned}$$

The answer: $a_n = -\frac{12}{49}\cdot 2^n + \frac{6}{7}\cdot(n+1)2^n - \frac{30}{49}\cdot 5^n$.

Exercise 1. Let $a_0 = 1$, and $a_n = a_{n-1} + n$ for $n \geq 1$. Use generating functions to find a formula for a_n. [1]

Exercise 2. Let $a_0 = 0$, and $a_n = 3a_{n-1} + 7$ for $n \geq 1$. Use generating functions to find a formula for a_n.

Exercise 3. For each of the following functions $G(n)$ use generating functions to find a formula for a_n, where $a_{n+2} - 6a_{n+1} - 7a_n = G(n)$.

(a) $G(n) = (-5)^n \cdot 24$, $a_0 = 3$, $a_1 = -1$.

(b) $G(n) = 12n^2 - 4n + 10$, $a_0 = 0$, $a_1 = -10$.

[1] The answer is $a_n = 1 + \binom{n+1}{2}$.