Math 232, Winter 2016 Boris Botvinnik

Summary on Lecture 5, January 11, 2016

The Method of Generating Functions.

There is another powerful technique to resolve recurrence relations. Let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence of real numbers. Then the series

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

is called a *generating function* for the sequence $\{a_i\}$.

Examples. **(1)** The function

$$f(x) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

is a generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, \ldots$.

(2) We notice that $1 - x^{n+1} = (1-x)(1+x+x^2+\cdots+x^n)$. This gives the generating function

$$\frac{1-x^{n+1}}{1-x}$$

for the sequence $1, 1, \ldots, 1, 0, 0, \ldots$.

(3) Similarly to the previous example, we notice that $1 = (1-x)(1+x+x^2+\cdots+x^n+\cdots)$. This gives the generating function

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

for the sequence $1, 1, \ldots, 1, \ldots$.

(4) Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx}\frac{1}{1-x} = \sum_{i=0}^{\infty} \frac{d}{dx} x^i$$

Since $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$, we obtain the identity:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

Thus the function $\frac{1}{(1-x)^2}$ is a generating function for the sequence $1, 2, 3, 4, \ldots, n, \ldots$. We also notice that the function

$$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots$$

is a generating function for the sequence $0, 1, 2, 3, 4, \ldots, n, \ldots$.

(5) We take one more derivative: Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots)$$

Since $\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{x+1}{(1-x)^3}$, we obtain the identity:

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots + n^2x^{n-1} + \cdots$$

Thus the function $\frac{x+1}{(1-x)^3}$ is a generating function for the sequence $1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots$. Then we see that the function

$$\frac{x(x+1)}{(1-x)^3} = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots + n^2x^n + \cdots$$

is a generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots$.

Before analyzing the examples, we need one more identity on generation function:

$$\frac{1}{(1+x)^k} = \sum_{n=0}^{\infty}(-1)^n\binom{k+n-1}{n}x^n \tag{1}$$

This could be verified using Taylor-Maclaurin decomposition of the function $\frac{1}{(1+x)^k}$.

Example 1. Let $a_0 = 1$, and $a_n - 3a_{n-1} = n$, $n \geq 1$. This is new type of recurrence relations: *non-homogeneous.* We write the first few terms:

$$\begin{array}{lcl} a_1 - 3a_0 & = & 1 \\ a_2 - 3a_1 & = & 2 \\ a_3 - 3a_2 & = & 3 \\ \dots & & \dots \\ a_n - 3a_{n-1} & = & n \\ \dots & & \dots \end{array}$$

We multiply the first equation by x, the second, by x^2, the third, by x^3 and so on. We get:

$$\begin{array}{lcl} a_1x - 3a_0x & = & x \\ a_2x^2 - 3a_1x^2 & = & x^2 \\ a_3x^3 - 3a_2x^3 & = & 3x^3 \\ \dots & & \dots \\ a_nx^n - 3a_{n-1}x^n & = & nx^n \\ \dots & & \dots \end{array}$$

We add the terms of the equations to get the identity:

$$\sum_{n=1}^{\infty} a_nx^n - 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \sum_{n=1}^{\infty} nx^n. \tag{2}$$

We denote $f(x) = \sum_{n=0}^{\infty} a_nx^n$, then

$$\sum_{n=1}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^n - a_0 = f(x) - 1,$$

$$\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = f(x).$$

Also, we recall from Example 4 (Lecture 4), we have

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \dots + nx^n + \dots = \frac{x}{(1-x)^2}$$

Then the identity (5) becomes

$$(f(x) - 1) - 3xf(x) = \frac{x}{(1-x)^2} \tag{3}$$

Then the identity (3) becomes

$$\begin{aligned}(f(x)-1)-3xf(x) &= -1+f(x)(1-3x)=\frac{x}{(1-x)^2}, \quad \text{or} \\ f(x)(1-3x) &= \frac{x}{(1-x)^2}+1, \qquad \text{or} \\ f(x) &= \frac{x}{(1-x)^2(1-3x)}+\frac{1}{(1-3x)}\end{aligned} \tag{4}$$

Now we would like to decompose the term $\frac{x}{(1-x)^2(1-3x)}$ as the sum using so called method of partial fractions:

$$\frac{x}{(1-x)^2(1-3x)}=\frac{A}{(1-x)}+\frac{B}{(1-x)^2}+\frac{C}{(1-3x)}$$

We obtain:

$$\begin{aligned} x &= A(1-x)(1-3x)+B(1-3x)+C(1-x)^2 \\ & 3Ax^2-4Ax+A+B-3Bx+C-2Cx+Cx^2 \\ & (3A+C)x^2+(-4A-3B-2C)x+(A+B+C)\end{aligned}$$

We get the system:

$$\left\{\begin{array}{lcl} 3A+C &=& 0 \\ -4A-3B-2C &=& 1 \\ A+B+C &=& 0 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} C &=& -3A \\ -4A-3B+6A &=& 1 \\ A+B-3A &=& 0 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} C &=& -3A \\ 2A-3B &=& 1 \\ -2A+B &=& 0 \end{array}\right.$$

The equations

$$\left\{\begin{array}{lcl} 2A-3B &=& 1 \\ -2A+B &=& 0 \end{array}\right.$$

give $-2B=1$, i.e. $B=-\frac{1}{2}$, and $A=\frac{1}{2}B=-\frac{1}{4}$. Then the equation $C=-3A$ gives $C=\frac{3}{4}$. We obtain:

$$\begin{aligned} f(x) &= -\frac{1}{4}\cdot\frac{1}{(1-x)}-\frac{1}{2}\cdot\frac{1}{(1-x)^2}+\frac{3}{4}\cdot\frac{1}{(1-3x)}+\frac{1}{(1-3x)} \\ &= -\frac{1}{4}\cdot\frac{1}{(1-x)}-\frac{1}{2}\cdot\frac{1}{(1-x)^2}+\frac{7}{4}\cdot\frac{1}{(1-3x)}\end{aligned}$$

Then we have

$$\begin{aligned} -\frac{1}{4}\cdot\frac{1}{(1-x)} &= -\frac{1}{4}(1+x+x^2+x^3+\cdots+x^n+\cdots) &&= \sum_{n=0}^{\infty}\left(-\frac{1}{4}x^n\right) \\ -\frac{1}{2}\cdot\frac{1}{(1-x)^2} &= -\frac{1}{2}\sum_{n=0}^{\infty}\left((-1)^n\binom{2+n-1}{n}(-x)^n\right) &&= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\binom{2+n-1}{n}x^n\right) \\ \frac{7}{4}\cdot\frac{1}{(1-3x)} &= \frac{7}{4}\sum_{n=0}^{\infty}(3x)^n &&= \sum_{n=0}^{\infty}\left(\frac{7\cdot 3^n}{4}x^n\right)\end{aligned}$$

We notice that $\binom{2+n-1}{n}=\binom{n+1}{n}=\binom{n+1}{1}=(n+1)$. Then we add all three series together to get

$$f(x)=\sum_{n=0}^{\infty}\left(-\frac{1}{4}-\frac{n+1}{2}+\frac{7\cdot 3^n}{4}\right)x^n=\sum_{n=0}^{\infty}\left(-\frac{3}{4}-\frac{n}{2}+\frac{7\cdot 3^n}{4}\right)x^n.$$

We obtain the answer: $a_n = -\frac{3}{4} - \frac{n}{2} + \frac{7 \cdot 3^n}{4} = \frac{7 \cdot 3^n - 3}{4} - \frac{n}{2}$.

Remark. We notice that for $n \geq 1$, the coefficient $\frac{3 \cdot 7(3^{n-1}-1)}{4} - \frac{n}{2}$ is always an integer. Prove it!

Example 2. Let $a_0 = 3$, $a_1 = 7$ and $a_{n+2} - 5a_{n+1} + 6a_n = 2$, $n \geq 2$. This is also a *non-homogeneous* new type of recurrence relation. We multiply the relation by x^{n+2}:

$$a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}$$

and we obtain the identity

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}. \tag{5}$$

We denote $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}x^{n+2} &= f(x) - a_0 - a_1 x &&= f(x) - 3 - 7x \\
\sum_{n=0}^{\infty} a_{n+1}x^{n+2} &= x(f(x) - a_0) &&= x(f(x) - 3) \\
\sum_{n=0}^{\infty} a_n x^{n+2} &= x^2 f(x) \\
\sum_{n=0}^{\infty} x^{n+2} &= \frac{x^2}{1-x}
\end{aligned}$$

Then the identity (5) turns to the equation for $f(x)$:

$$\begin{aligned}
(f(x) - 3 - 7x) - 5x(f(x) - 3) + 6x^2 f(x) &= \frac{2x^2}{1-x} \quad \text{or} \\
f(x)(1 - 5x + 6x^2) &= \frac{2x^2}{1-x} + 3 - 8x
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
f(x) &= \frac{(3-8x)(1-x) + 2x^2}{(1-x)(1-5x+6x^2)} \\
&= \frac{3 - 11x + 10x^2}{(1-x)(1-5x+6x^2)} \\
&= \frac{(3-5x)(1-2x)}{(1-x)(1-2x)(1-3x)} = \frac{3-5x}{(1-x)(1-3x)}
\end{aligned}$$

Here we used that $3 - 11x + 10x^2 = (3-5x)(1-2x)$ and $1 - 5x + 6x^2 = (1-2x)(1-3x)$. Now we write

$$\frac{3-5x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x} = \frac{A - 3Ax + B - Bx}{(1-x)(1-3x)} = \frac{(A+B) - (3A+B)x}{(1-x)(1-3x)}$$

which yields the system:

$$\left\{ \begin{aligned} A + B &= 3 \\ 3A + B &= 5 \end{aligned} \right. \quad \text{or} \quad \left\{ \begin{aligned} A &= 1 \\ B &= 2 \end{aligned} \right.$$

This gives the generating function:

$$f(x) = \frac{1}{1-x} + \frac{2}{1-3x} = \sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (1 + 2 \cdot 3^n) x^n.$$

We obtain the solution: $a_n = 1 + 2 \cdot 3^n\,,\ n \geq 0\,.$