

Summary on Lecture 4, January 8, 2016

Second Order Recurrence Relations (continuation)

Example: legal arithmetic expressions without parenthesis. In most computing languages, it important to use “legal arithmetic expressions without parenthesis”. These expressions are made up out of the digits $0, 1, \dots, 9$ and binary symbols $+, *, /$. For example, the expressions $7 + 8$, $5 + 7 * 3$, $33 * 7 + 4 + 6 * 4$ are legal expressions, and the expressions $/7 + 8$, $5 + 7 * 3 +$, $33 * 7 + /4 + 6 * 4$ are not.

We denote by a_n the number of legal expressions of length n . Then $a_1 = 10$ since the only legal expressions of length 1 are the digits $0, 1, \dots, 9$. Then $a_2 = 100$ which accounts for the expressions $00, 01, \dots, 99$.

Let $n \geq 3$. We observe:

(1) Let x be an arithmetic legal expression of $(n - 1)$ symbols. Then the last symbol must be a digit. We add one more digit to the right of x and obtain $10x$ more legal expressions of the length n .

(2) Let y be an arithmetic legal expression of $(n - 2)$ symbols. Then we can add to the right of y one of the following 29 2-symbol expressions: $+0, +1, \dots, +9, *0, *1, \dots, *9, /1, \dots, /9$ (no division by 0 is allowed).

We obtain the recurrence relation: $a_0 = 10, a_1 = 10, a_n = 10a_{n-1} + 29a_{n-2}$ for $n \geq 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 10, a_1 = 10, a_n = 10a_{n-1} + 29a_{n-2}, n \geq 3$.

Example. We would like to find a number of binary sequences of the length n without any consecutive 0's.

Let a_n denote the number of such sequences of length $n \geq 1$. Clearly, if $n = 1$, we have $0, 1$, i.e., $a_1 = 2$, if $n = 2$, we have the sequences $01, 10, 11$, i.e., $a_2 = 3$.

Let $n \geq 3$. Let $x_1 \cdots x_{n-2}x_{n-1}x_n$ be a sequence like that. There are two cases:

(1) The last symbol $x_n = 1$. Then the sequence $x_1 \cdots x_{n-2}x_{n-1}$ has no consecutive 0's.

(2) The last symbol $x_n = 0$. Then $x_{n-1} = 1$, and the sequence $x_1 \cdots x_{n-2}$ has no consecutive 0's.

Thus we conclude that $a_n = a_{n-1} + a_{n-2}$. Also we notice that the initial conditions $a_1 = 2, a_2 = 3$ could be replaced by $a_0 = 1, a_1 = 2$. Then $a_2 = a_1 + a_0 = 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 1, a_1 = 2, a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

The case of complex roots. Let $z = x + iy \in \mathbf{C}$ be a complex number. Then we let $|z| = \sqrt{x^2 + y^2}$, and we can write z as

$$z = |z|(\cos \theta + i \sin \theta), \quad \cos \theta = \frac{x}{|z|}, \quad \sin \theta = \frac{y}{|z|}.$$

There is a standard notation $e^{i\theta} := \cos \theta + i \sin \theta$. There is a remarkable formula (DeMoivre Theorem):

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \quad \text{or} \quad (e^{i\theta})^n = e^{in\theta}$$

We prove it by induction. Clearly this formula holds for $n = 1$. Assume it holds for $n = k$. Then we have:

$$\begin{aligned} (\cos \theta + i \sin \theta)^{k+1} &= (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \\ &= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta) \\ &= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\cos k\theta \sin \theta + \sin k\theta \cos \theta) \\ &= \cos(k+1)\theta + i \sin(k+1)\theta. \end{aligned}$$

Here we used the formulas:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Example. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} - 2a_{n-2}$. Then again, we are looking for a solution as $a_n = cr^n$, $c \neq 0$. We have substitute $a_n = cr^n$ to our recurrence relation:

$$cr^n = 2cr^{n-1} - 2cr^{n-2} \quad \text{or} \quad r^2 - 2r + 2 = 0.$$

We find the solutions of the characteristic equation:

$$r_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm i.$$

Then we have:

$$r_1 = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right),$$

$$r_2 = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right).$$

Now we are looking for a solution in the form $a_n = c_1r_1^n + c_2r_2^n$. We notice the following:

$$\begin{aligned} a_n &= c_1(1+i)^n + c_2(1-i)^n \\ &= c_1\left(\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right)^n + c_2\left(\sqrt{2}\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right)\right)^n \\ &= c_1\left(\sqrt{2}\right)^n \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4}\right) + c_2\left(\sqrt{2}\right)^n \left(\cos \frac{n\pi}{4} - i \sin \frac{n\pi}{4}\right) \\ &= \left(\sqrt{2}\right)^n \left(K_1 \cos \frac{n\pi}{4} + K_2 \sin \frac{n\pi}{4}\right), \end{aligned}$$

where $K_1 = c_1 + c_2$, $K_2 = i(c_1 - c_2)$. Clearly we would like to find **real values** of K_1 and K_2 . We substitute $n = 0$ and $n = 1$ to get the system:

$$\begin{cases} K_1 \cos 0 + K_2 \sin 0 & = a_0 = 1 \\ \sqrt{2}(K_1 \cos \frac{\pi}{4} + K_2 \sin \frac{\pi}{4}) & = a_1 = 2 \end{cases} \quad \text{or} \quad \begin{cases} K_1 & = 1 \\ K_1 + K_2 & = 2 \end{cases} \quad \text{or} \quad \begin{cases} K_1 & = 1 \\ K_2 & = 1 \end{cases}$$

We obtain the answer:

$$a_n = \left(\sqrt{2}\right)^n \left(\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}\right)$$

General Case. Now we assume that we have a second order recurrence relation, i.e. a_0, a_1 , are given and $a_n = Aa_{n-1} + Ba_{n-2}$, $n \geq 2$, where the coefficients A and B are real numbers (in fact, they are integers in all our examples). Then the characteristic equation is given as $r^2 - Ar - B = 0$. Assume that the roots r_1, r_2 are complex. Obviously, it means that r_1 and r_2 are conjugate, i.e., we can write $r_1 = \rho(\cos \theta + i \sin \theta)$, and $r_2 = \rho(\cos \theta - i \sin \theta)$. Thus we may look for a solution for a_n in the form:

$$\begin{aligned} a_n &= c_1r_1^n + c_2r_2^n \\ &= c_1\rho^n(\cos n\theta + i \sin n\theta) + c_2\rho^n(\cos n\theta - i \sin n\theta) \\ &= \rho^n((c_1 + c_2) \cos n\theta + i(c_1 - c_2) \sin n\theta) \\ &= \rho^n(K_1 \cos n\theta + K_2 \sin n\theta). \end{aligned}$$

Here $K_1 = (c_1 + c_2)$ and $K_2 = i(c_1 - c_2)$. In particular, it means that the expression $a_n = \rho^n(K_1 \cos n\theta + K_2 \sin n\theta)$ satisfies the recurrence relation we started with. We also notice that K_1 and K_2 are assumed to be real. Since a_0 and a_1 are given, we find them by substituting $n = 0$ and $n = 1$:

$$\begin{cases} K_1 \cos 0 + K_2 \sin 0 & = a_0 \\ \rho(K_1 \cos \theta + K_2 \sin \theta) & = a_1 \end{cases} \quad \text{or} \quad \begin{cases} K_1 & = a_0 \\ \rho(K_1 \cos \theta + K_2 \sin \theta) & = a_1 \end{cases}$$

We notice that the system always has a solution provided $\rho \sin \theta \neq 0$ for arbitrary initial coefficients a_0 and a_1 . On the other hand, the condition $\rho \sin \theta = 0$ means that either $\rho = 0$ or $\sin \theta = 0$. Each of those imply that the roots r_1 r_2 are real.

We summarize the above discussion:

Theorem 3. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero real constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two complex roots

$$r_1 = \rho(\cos \theta + i \sin \theta), \quad r_2 = \rho(\cos \theta - i \sin \theta),$$

Then $a_n = \rho^n(K_1 \cos n\theta + K_2 \sin n\theta)$, where the coefficients K_1, K_2 are determined by solving the system

$$\begin{cases} K_1 & = a_0 \\ \rho(K_1 \cos \theta + K_2 \sin \theta) & = a_1 \end{cases}$$