Summary on Lecture 19, February 19, 2016

Digraphs and shortest paths

Let $G=(V,E)$ be a graph or digraph. We say that G is *weighted* if we are given a function $\mathsf{wt}: E \to (0,\infty)$. In other words, each edge $e\in E$ is given a positive weight $\mathsf{wt}(e)$. It is convenient to have a convention that if $v,v'\in V(G)$ are not connected by an edge, then a *virtual edge* (v,v') has weight ∞. We also accept a convention that a *virtual edge* (v,v) has zero weight. Then once we have a weighted graph, it makes sense to determine a *shortest distance* $d(v,v')$ from a vertex v to a vertex v'.

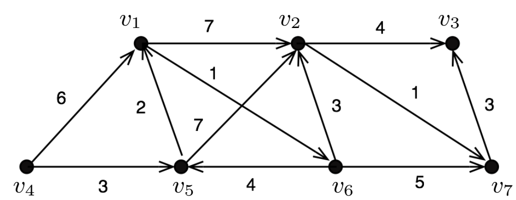

Fig. 1. Weighted digraph G_1.

Example. Consider the digraph G_1 in Fig. 1. It is easy to check that the shortest distance (weight) from v_4 to v_1 is 5 when we take the route $v_4\to v_5\to v_1$, and it is shorter than a "direct" shot $v_4\to v_1$.

Now the "shortest" paths $v_4\to v_1\to v_2\to v_3$ and $v_4\to v_5\to v_2\to v_3$ from v_4 to v_3 have weights $6+7+4=17$ and $3+7+4=14$, respectively, but the "longer" path $v_4\to v_5\to v_1\to v_6\to v_2\to v_7\to v_4$ has weight $3+2+1+3+1+3=13$, which is less than either of these. Thus length is not directly related to weight. $\square$

There are special vertices in G_1, namely, v_4 is a *source*, and v_3 is a *sink*. Indeed, v_4 has only *outbounded* edges, and v_3 only *inbounded* edges.

In the case when a weighted digraph $G=(V,E)$ with a only one source v_0 and one sink v_* and has no loops and parallel edges), it is called a *scheduling network*. The objective here is to find a shortest path (i.e., with minimal weight) from a source to a sink. This is known as *scheduling problem*. We assume that a weight of every edge is positive. In fact, we resolve more general problem, namely, for each vertex $v_0\in V$ we determine

(i) the distance $d(v_0,v)$ for every $v\in V$;

(ii) a shortest path from v_0 to v if such a path exists.

We subdivide the set of vertices V into two subsets: $V=S\cup\bar{S}$, where $v_0\in S$, and $\bar{S}=V\setminus S$, so that $S\cap\bar{S}=\emptyset$. Then we define the distance $d(v_0,\bar{S})$:

$$d(v_0,\bar{S})=\min\{d(v,\bar{v})\mid \bar{v}\in\bar{S}\ \}.$$

If the distance $d(v_0,\bar{S})$ is finite, then there exists at least one vertex $\bar{v}_*\in\bar{S}$ such that $d(v_0,\bar{S})=d(v_0,\bar{v}_*)$. We choose a shortest path P from v_0 to $\bar{v}_*$:

$$P: v_0\to v_1\to v_2\to\cdots\to v_k\to v_{k+1}=\bar{v}_*$$

Now we make the following observations:

(1) The vertices $v_0,v_1,\dots v_k$ are in S. Indeed, if v_i would be in $\bar{S}$ for some $i=1,\dots,k$, then the path $P_i: v_0\to v_1\to v_2\to\cdots\to v_i$ would be shorter than P.

(ii) The path $P_i: v_0\to v_1\to v_2\to\cdots\to v_i$ is the shortest path in G from v_0 to v_i for each $i=1,\dots,k$. Indeed, if there would be a shorter path P_i' from v_0 to v_i, then we would find a path P_i' composed with the path $v_i\to\cdots\to v_{k+1}=\bar{v}_*$ which is shorter than P.

Lemma 1. $d(v_0, \bar{S}) = \min\{\ d(v_0, u) + \mathsf{wt}(u, \bar{v}) \mid u \in S,\ \ \bar{v} \in \bar{S}\ \}$.

Then if the minimum occurs when $u = u_* \in S$ and $\bar{v} = \bar{v}_* \in \bar{S}$, then we have that

$$d(v_0, \bar{S}) = d(v_0, u_*) + \mathsf{wt}(u, \bar{v}_*). \tag{1}$$

We use the formula (1) to explain the idea of the **Dijkstra's Shortest Path Algorithm**. It goes as follows. Let $S_0 = \{v_0\}$, and $\bar{S}_0 = V \setminus S_0$. We find $d(v_0, \bar{S}_0)$:

$$d(v_0, \bar{S}_0) = \min\{\ \mathsf{wt}(v_0, \bar{v}) \mid \bar{v} \in \bar{S}_0\}.$$

Let $v_1 \in \bar{S}_0$ be such that $d(v_0, \bar{S}_0) = d(v_0, v_1)$. Then we define $S_1 = S_0 \cup \{v_1\}$. Then we find $d(v_0, \bar{S}_1)$:

$$d(v_0, \bar{S}_1) = \min\{\ d(v_0, v_1) + \mathsf{wt}(v_1, \bar{v}) \mid \bar{v} \in \bar{S}_1\ \}.$$

We find $v_2 \in \bar{S}_1$ such that $d(v_0, \bar{S}_1) = d(v_0, v_1) + \mathsf{wt}(v_1, v_2)$. If we proceed in such a way, we construct a set $S_i = \{v_0, v_1, \ldots, v_i\}$, and find the distance $d(v_0, \bar{S}_i)$:

$$d(v_0, \bar{S}_i) = \min\{\ d(v_0, v_i) + \mathsf{wt}(v_i, \bar{v}) \mid \bar{v} \in \bar{S}_i\ \},$$

and find the next vertex $v_{i+1} \in \bar{S}_i$ such that $d(v_0, \bar{S}_i) = \min\{\ d(v_0, v_i) + \mathsf{wt}(v_i, v_{i+1})$. The algorithm will stop if either $d(v_0, \bar{S}_i) = \infty$, or $|S| = |V|$. Fig. 2 shows the resulting path for the graph G_1 as above.

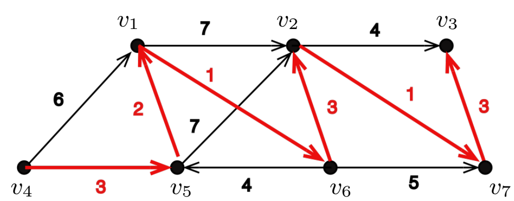

Fig. 2. A shortest path from v_4 to v_3.

Next time we analyze the Dijkstra's Shortest Path Algorithm in detail.