Summary on Lecture 12, February 3, 2016

## More on Chromatic Polynomials

Let $G=(V,E)$ be a graph, and $e$ be its edge with vertices $a$ and $b$. We denote by $G_e$ the graph which is obtained by removing the edge $e$. Let $G'_e$ be a graph which is obtained from $G_e$ by identifying the vertices $a$ and $b$, see Fig. 3. Last time we proved the following Theorem:

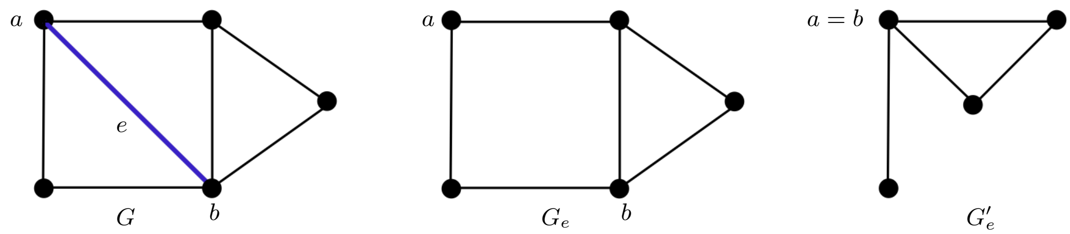


Fig. 3. The graphs $G$, $G_e$ and $G'_e$

**Theorem 1.** *Let $G=(V,E)$ be a connected graph, and $e\in E$. Then*

$$P(G_e,\lambda)=P(G,\lambda)+P(G'_e,\lambda).$$

**Proof.** Let $e=\{a,b\}$. Consider the value $P(G_e,\lambda)$. There are two possibilities here: either the vertices $a$ and $b$ have the same color or not. If they are of different colors, then it corresponds to a proper coloring of $G$. If they are the same, then it corresponds to a proper coloring of $G'_e$. □

**Lemma 1.** *Let $T$ be a tree with $n$ vertices. Then $P(T,\lambda)=\lambda(\lambda-1)^{n-1}$.*

**Proof.** Induction on $n$. If $n=1$, then obviously $P(T,\lambda)=\lambda$. Let $n>1$. We find an edge $e$ such that $e=\{a,b\}$, where $a$ is a leaf. Then $T_e$ is a disjoint union of a tree on $(n-1)$ vertices and a single vertex, and $T'_e$ is a tree on $(n-1)$ vertices, see Fig. 4

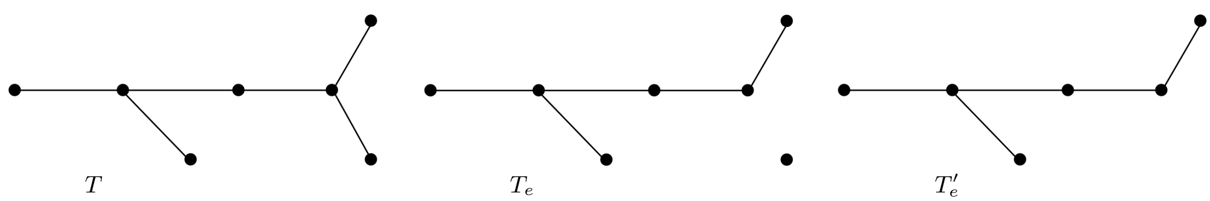


Fig. 4. The graphs $T$, $T_e$ and $T'_e$

By induction, we have that $P(\lambda,T'_e)=\lambda(\lambda-1)^{n-2}$, and $P(\lambda,T_e)=\lambda(\lambda-1)^{n-2}\cdot\lambda=\lambda^2(\lambda-1)^{n-2}$. Then

$$P(\lambda,T)=P(\lambda,T_e)-P(\lambda,T'_e)=\lambda^2(\lambda-1)^{n-2}-\lambda(\lambda-1)^{n-2}=\lambda(\lambda-1)^{n-1}.$$

This completes the induction. □

**Lemma 2.** *Let $C_n$ be a cycle on $n$ vertices. Then $P(C_n,\lambda)=(\lambda-1)^n+(-1)^n(\lambda-1)$.*

**Proof.** Induction on $n$. If $n=3$, $C_3=K_3$, so we check

$$P(K_3,\lambda)=\lambda(\lambda-1)(\lambda-2)=(\lambda-1)^2+(-1)^2(\lambda-1)=(\lambda-1)(\lambda-1-1).$$

Let $n>3$, and $e$ be an edge of $C_n$. Then $(C_n)_e$ is just a path on $n$ vertices, and $(C_n)'_e=C_{n-1}$ is just a cycle on $(n-1)$ vertices. We have

$$\begin{aligned} P(C_n,\lambda) &= P((C_n)_e,\lambda)-P(C_{n-1},\lambda) \\ &= \lambda(\lambda-1)^{n-1}-(\lambda-1)^{n-1}-(-1)^{n-1}(\lambda-1) \\ &= (\lambda-1)^n+(-1)^n(\lambda-1). \end{aligned}$$

This completes the induction. □

**Remark.** We notice that $P(C_n, 1) = 0$, and $P(C_n, 2) = 1 + (-1)^n$. Hence $P(C_n, 2) = 2$ if $n$ is even, and $P(C_n, 2) = 0$ if $n$ is odd. Let $n = 2k + 1$, then $P(C_{2k+1}, 3) = 2^{2k+1} - 2 = 2(2^{2k} - 1)$. We conclude that $\chi(C_n) = 2$ if $n$ is even and $\chi(C_n) = 2(2^{2k} - 1)$ if $n = 2k + 1$.

We define a *wheel on* $(n+1)$ *vertices* $W_{n+1}$ by taking a cycle on $n$ vertices and connecting each vertex of a cycle to one more $(n+1)$ the vertex.

**Lemma 3.** *Let* $W_{n+1}$ *be a wheel on* $(n+1)$ *vertices. Then* $P(W_{n+1}, \lambda) = \lambda(\lambda - 2)^n - (-1)^n \lambda(\lambda - 2)$.

**Proof.** We assign an arbitrary color to the centeral vertex, then we can use $(\lambda - 1)$ colors for the remaining vertices. We obtain:

$$P(W_{n+1}, \lambda) = \lambda \left[(\lambda - 1 - 1)^n - (-1)^n(\lambda - 1 - 1)\right] = \lambda(\lambda - 2)^n - (-1)^n \lambda(\lambda - 2).$$

This proves Lemma 3. □

We make two observations about the chromatic polynomial. Let $G = (V, E)$, and

$$P(G, \lambda) = a_0 + a_1\lambda + \cdots + a_d\lambda^d.$$

(1) The coefficient $a_0 = 0$. Indeed, $P(G, 0) = a_0$ and at the same time is a number of proper colorings of $G$ with $0$ colors, i.e., $a_0 = 0$.

(2) Assume that $|E| > 0$. Then $a_1 + a_2 + \cdots + a_d = P(G, 1) = 0$. Indeed, if $G$ contains an edge, then we cannot give proper colorings of $G$ with $1$ color.

**Lemma 4.** *Let* $G_1$ *and* $G_2$ *share a single vertex, i.e.,* $V(G_1) \cap V(G_2) = \{v\}$. *Then*

$$P(G_1 \cup G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda}.$$

**Proof.** Let us give a coloring to $G_2$. Then the vertex $v$ receives some color $\lambda_0$. Then we count how many ways to color $G_1$ so that $v$ is colored with given color $\lambda_0$. Let $Q(G_1; v, \lambda, \lambda_0)$ be the result. Because of the symmetry with respect to colors, $Q(G_1; v, \lambda, \lambda_0)$ is the same for any choice of $\lambda_0$. On the other hand, by taking every value of $\lambda_0$, we count all colorings of the graph $G_1$. We conclude that $Q(G_1; v, \lambda, \lambda_0)\lambda = P(G_1, \lambda)$, or

$$Q(G_1; v, \lambda, \lambda_0) = \frac{P(G_1, \lambda)}{\lambda}$$

We obtain that all colorings of $G_1 \cup G_2$ are given by

$$\frac{P(G_1, \lambda)}{\lambda} \cdot P(G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda}.$$

This completes the proof. □

**Lemma 5.** *Let* $G_1$ *and* $G_2$ *be such that* $G_1 \cap G_2 = K_n$. *Then*

$$P(G_1 \cup G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{P(K_n, \lambda)} = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}.$$

(Recall that $\lambda^{(n)} := \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$.)

**Exercise.** Prove Lemma 5.