Summary on Lecture 10, January 25, 2016

Hamiltonian Cycle and Hamiltonian Path

Euler's Theorem (Theorem 3 from Lecture 8) tells us which graphs have closed paths that use each edge exactly once, and the algorithm **EulerCircuit** gives a way to construct the paths when they exist.

In contrast, much less is known about graphs with paths that use each vertex exactly once. The Irish mathematician **Sir William Hamilton** (1805–1865) was one of the first to study such graphs.

Let $G = (V, E)$ be a graph. We say that a cycle in G is a *Hamiltonian cycle* if it visits every vertex of G (only once), and a path is called a *Hamilton path* if it visits every vertex of the graph exactly once. A graph with a Hamilton cycle is called a *Hamiltonian graph*.

In particular, Hamilton invented a game known now as *Hamilton's puzzle*. The game's objectinve is finding a Hamiltonian cycle along the edges of a dodecahedron, i.e., such that every vertex is visited a single time, and the ending point is the same as the starting point. Here is a solution of this puzzle:

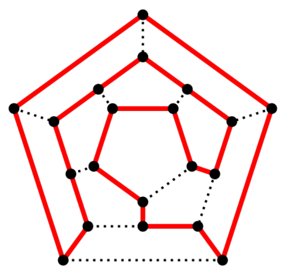

Fig. 10. Hamilton's puzzle[1]

The problem of finding a Hamiltonian cycle sounds very similar to the one of finding an Euler circuit. However, this problem is much harder, and there are still no complete solution. We prove several results on the existence of a Hamiltonian cycle. Here are examples of two Hamiltonian graphs (on the left) and two non-Hamiltonian graphs (on the right).

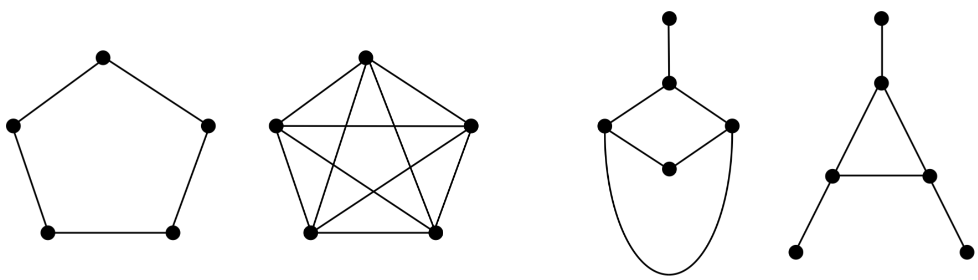

Fig. 11. Hamiltonian and not Hamiltonian graphs

Lemma 1. *The complete graph K_n has a Hamiltonian cycle.*

Proof. Let $V(K_n) = (v_1, \ldots, v_n)$. Recall that every two distinct vertices v_i, v_j are connected in K_n by a single edge $e_{i,j}$. Clearly the path $P = e_{1,2}e_{2,3}\cdots e_{n-1,n}e_{n,1}$ is a Hamiltonian cycle. □

[1]These pictures are taken from Wikipedia

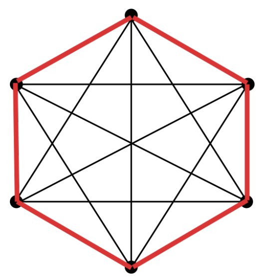

Fig. 12. K_n is a Hamiltonian graph

I would like to start with a result giving sufficient conditions for a graph G to be Hamiltonian. If G has no loop an multiple edges, and, in addition, $|V(G)| \geq 3$, we can always think of G as a subgraph of the complete graph K_n. We will use this fact below.

Theorem 1. *Let $G = (V, E)$ be a graph without loops or multiple edges with $|V| = n \geq 3$. Assume that $\deg v + \deg v' \geq n$ for every two distinct vertices $v, v' \in V$ which are not connected by an edge. Then G has a Hamiltonian cycle.*

Proof. Assume Theorem 1 is false for some $n \geq 3$. Then there exists a graph $G = (V, E)$ which satisfies the conditions of Theorem 1, however, G is not a Hamiltonian graph. We consider the set of such counterexamples

$$\mathcal{G} = \left\{ G = (V, E) \;\middle|\; V(G) = n, \quad \begin{array}{c} G \text{ satisfies the conditions of Theorem 1 and} \\ G \text{ is not a Hamiltonian graph} \end{array} \right\} \neq \emptyset$$

Recall that every $G \in \mathcal{G}$ is a subgraph of K_n. Since any such graph G does not have loops or multiple edges, $|E(G)| < |E(K_n)| = \binom{n}{2}$. In particular, for given n, there exist only finite number of counterexamples. We choose a graph with maximal number of edges, and we can assume that the above graph G is such a counterexample in the first place. In particular, it means that if we add one more edge e, then the graph $G \cup \{e\}$ becomes Hamiltonian, and there exists a Hamiltonian cycle. We find such a cycle and then delete new edge e. This gives us a Hamiltonian path in the graph G.

Let us list the vertices of G in the order they were visited in the Hamiltonian path: $(v_1, v_2, \ldots, v_n)$. We notice that the vertices v_1 and v_n are not connected by an edge, otherwise the graph G would be Hamiltonian. Thus $\deg v_1 + \deg v_n \geq n$ by hypothesis.

We define two sets of vertices:

$$S_1 = \{ \ i \mid \{v_1, v_i\} \in E(G) \ \}, \quad S_n = \{ \ j \mid \{v_{j-1}, v_n\} \in E(G) \ \}.$$

By construction and the hypothesis, $S_1 \cup S_n \subset \{2, 3, \ldots, n-1\}$, and $|S_1| = \deg v_1$, and $|S_n| = \deg v_n$. We have that $|S_1| + |S_n| \geq n$, and $|S_1 \cup S_n| \leq n-1$. We conclude that $S_1 \cap S_n$ is not empty. Indeed, assume $S_1 \cap S_n = \emptyset$, then

$$|S_1 \cup S_n| = |S_1| + |S_n|,$$

however, we know $|S_1 \cup S_n| \leq n-1$ and $|S_1| + |S_n| \geq n$. Let $i_0 \in S_1 \cup S_n$. Then there exists an edge $\{v_1, v_{i_0}\}$ and an edge $\{v_{i_0 - 1}, v_n\}$, as it is shown below (in red):

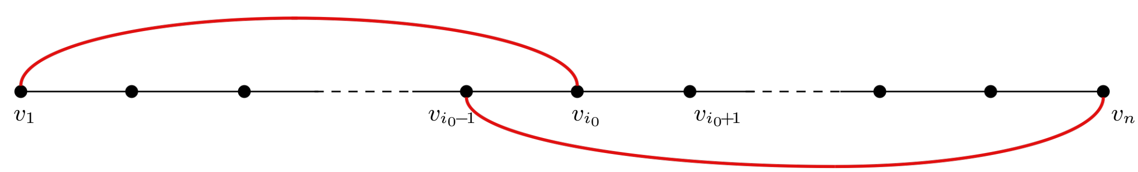

Fig. 13. A Hamiltonian cycle in G

Then the path $(v_1, \ldots, v_{i_0-1}, v_n, v_{n-1}, \ldots, v_{i_0+1}, v_{i_0}, v_1)$ is a Hamiltonian cycle. □

Theorem 2. *Let $G = (V, E)$ be a graph without loops or multiple edges with $|V| = n \geq 3$. Assume that $\deg v \geq \frac{n}{2}$ for every vertex $v \in V$. Then G has a Hamiltonian cycle.*

Proof. By assumption, $\deg v \geq \frac{n}{2}$ for every vertex $v \in V$. Thus $\deg v + \deg v' \geq n$ for every distinct vertices $v, v' \in V(G)$. In particular, the hypothesis of Theorem 1 are satisfied. Then G is a Hamiltonian graph. □

Theorem 3. *Let $G = (V, E)$ be a graph without loops or multiple edges with $|V| = n \geq 3$. Assume that G has at least $\frac{1}{2}(n-1)(n-2)+2$ edges. Then G has a Hamiltonian cycle.*

Proof. We notice that

$$|E(G)| \geq \frac{1}{2}(n-1)(n-2) + 2 = \binom{n-1}{2} + 2.$$

Since G is a subgraph of K_n, we have two cases:

(1) The graph G coincides with K_n.

(2) There are two vertices $v, v' \in V(G)$ which are not connected by an edge.

In the case (1) there is nothing to prove. In the case (2), we remove the vertices v, v' from G and all edges which have vertices v or v'. We denote by G' the resulting graph. Since G' is a subgraph of K_{n-2}, we have that

$$\binom{n-2}{2} = |E(K_{n-2})| \geq |E(G')| \geq \frac{1}{2}(n-1)(n-2) + 2 - \deg v - \deg v'.$$

Now we have:

$$\begin{aligned}
\deg v + \deg v' &\geq \binom{n-1}{2} - \binom{n-2}{2} + 2 \\
&= \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-2)(n-3) + 2 \\
&= \frac{1}{2}(n-2)(n-1-n+3) + 2 \\
&= \frac{1}{2}(n-2) \cdot 2 + 2 = n.
\end{aligned}$$

Clearly, this is true for all vertices $v, v' \in V(G)$ which are not connected by an edge. Then G satisfies the hypothesis of Theorem 1. □

Remark. Theorems 1, 2, and 3 are somewhat unsatisfactory in two ways. Not only are their sufficient conditions not necessary, but the theorems also give no guidance for finding a Hamilton circuit when one is guaranteed to exist. We emphasize that no efficient algorithm is known for finding Hamilton cycles. On the positive side, a Hamiltonian graph must certainly be connected, so all three theorems give sufficient conditions for a graph to be connected.

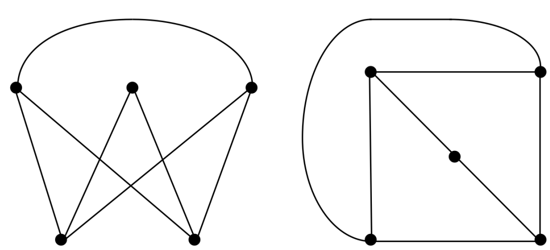

Fig. 14. Hamiltonian graphs for which Theorems do not work so well