Summary on Lecture 5, January 11, 2015

The Method of Generating Functions: Examples.

Before analyzing the examples, we need one more identity on generation function:

$$\frac{1}{(1+x)^k} = \sum_{n=0}^{\infty}(-1)^n \binom{k+n-1}{r} x^n \tag{1}$$

This could be verified using Taylor-Maclaurin decomposition of the function $\frac{1}{(1+x)^n}$.

Example 1. Let $a_0 = 1$, and $a_n - 3a_{n-1} = n$, $n \geq 1$. This is new type of recurrence relations: *non-homogeneous.* We write the first few terms:

$$\begin{array}{lcl} a_1 - 3a_0 & = & 1 \\ a_2 - 3a_1 & = & 2 \\ a_3 - 3a_2 & = & 3 \\ \dots & & \dots \\ a_n - 3a_{n-1} & = & n \\ \dots & & \dots \end{array}$$

We multiply the first equation by x, the second, by x^2, the third, by x^3 and so on. We get:

$$\begin{array}{lcl} a_1x - 3a_0x & = & x \\ a_2x^2 - 3a_1x^2 & = & x^2 \\ a_3x^3 - 3a_2x^3 & = & 3x^3 \\ \dots & & \dots \\ a_nx^n - 3a_{n-1}x^n & = & nx^n \\ \dots & & \dots \end{array}$$

We add the terms of the equations to get the identity:

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \sum_{n=1}^{\infty} nx^n. \tag{2}$$

We denote $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - a_0 = f(x) - 1,$$

$$\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = f(x).$$

Also, we recall from Example 4 (Lecture 4), we have

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots = \frac{x}{(1-x)^2}$$

Then the identity (2) becomes

$$(f(x) - 1) - 3f(x) = \frac{x}{(1-x)^2} \tag{3}$$

Then the identity (3) becomes

$$\begin{aligned}
(f(x)-1)-3xf(x) &= -1+f(x)(1-3x)=\frac{x}{(1-x)^2}, \quad \text{or}\\
f(x)(1-3x) &= \frac{x}{(1-x)^2}+1, \qquad \text{or}\\
f(x) &= \frac{x}{(1-x)^2(1-3x)}+\frac{1}{(1-3x)}
\end{aligned} \tag{4}$$

Now we would like to decompose the term $\frac{x}{(1-x)^2(1-3x)}$ as the sum using so called method of partial fractions:

$$\frac{x}{(1-x)^2(1-3x)}=\frac{A}{(1-x)}+\frac{B}{(1-x)^2}+\frac{C}{(1-3x)}$$

We obtain:

$$\begin{aligned}
x &= A(1-x)(1-3x)+B(1-3x)+C(1-x)^2\\
& \quad 3Ax^2-4Ax+A+B-3Bx+C-2Cx+Cx^2\\
& \quad (3A+C)x^2+(-4A-3B-2C)x+(A+B+C)
\end{aligned}$$

We get the system:

$$\left\{\begin{array}{lcl} 3A+C &=& 0\\ -4A-3B-2C &=& 1\\ A+B+C &=& 0\end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} C &=& -3A\\ -4A-3B+6A &=& 1\\ A+B-3A &=& 0\end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} C &=& -3A\\ 2A-3B &=& 1\\ -2A+B &=& 0\end{array}\right.$$

The equations

$$\left\{\begin{array}{lcl} 2A-3B &=& 1\\ -2A+B &=& 0\end{array}\right.$$

give $-2B=1$, i.e. $B=-\frac{1}{2}$, and $A=\frac{1}{2}B=-\frac{1}{4}$. Then the equation $C=-3A$ gives $C=\frac{3}{4}$. We obtain:

$$\begin{aligned}
f(x) &= -\frac{1}{4}\cdot\frac{1}{(1-x)}-\frac{1}{2}\cdot\frac{1}{(1-x)^2}+\frac{3}{4}\cdot\frac{1}{(1-3x)}+\frac{1}{(1-3x)}\\
&= -\frac{1}{4}\cdot\frac{1}{(1-x)}-\frac{1}{2}\cdot\frac{1}{(1-x)^2}+\frac{7}{4}\cdot\frac{1}{(1-3x)}
\end{aligned}$$

Then we have

$$\begin{aligned}
-\frac{1}{4}\cdot\frac{1}{(1-x)} &= -\frac{1}{4}(1+x+x^2+x^3+\cdots+x^n+\cdots) &&= \sum_{n=0}^{\infty}\left(-\frac{1}{4}x^n\right)\\
-\frac{1}{2}\cdot\frac{1}{(1-x)^2} &= -\frac{1}{2}\sum_{n=0}^{\infty}\left((-1)^n\binom{2+n-1}{n}(-x)^n\right) &&= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\binom{2+n-1}{n}x^n\right)\\
\frac{7}{4}\cdot\frac{1}{(1-3x)} &= \frac{7}{4}\sum_{n=0}^{\infty}(3x)^n &&= \sum_{n=0}^{\infty}\left(\frac{7\cdot 3^n}{4}x^n\right)
\end{aligned}$$

We notice that $\binom{2+n-1}{n}=\binom{n+1}{n}=\binom{n+1}{1}=(n+1)$. Then we add all three series together to get

$$f(x)=\sum_{n=0}^{\infty}\left(-\frac{1}{4}-\frac{n+1}{2}+\frac{7\cdot 3^n}{4}\right)x^n=\sum_{n=0}^{\infty}\left(-\frac{3}{4}-\frac{n}{2}+\frac{7\cdot 3^n}{4}\right)x^n.$$

We obtain the answer: $a_n=-\frac{3}{4}-\frac{n}{2}+\frac{7\cdot 3^n}{4}=\frac{7\cdot 3^n-3}{4}-\frac{n}{2}$. [1]

[1] We notice that $n\geq 1$, the coefficient $\frac{3\cdot 7(3^{n-1}-1)}{4}-\frac{n}{2}$ is always an integer. Prove it!