Summary on Lecture 4, January 9, 2015

## Second Order Recurrence Relations: the case of complex roots.

Again, we assume that we have a second order recurrence relation, i.e. $a_0$, $a_1$, are given and $a_n = Aa_{n-1}+Ba_{n-2}$, $n \geq 3$, where the coefficients $A$ and $B$ are real numbers (in fact, they are integers in all our examples). Then the characteristic equation is given as $r^2 - Ar - B = 0$. Assume that the roots $r_1$, $r_2$ are complex. Obviously, it means that $r_1$ and $r_2$ are conjugate, i.e., we can write $r_1 = \rho(\cos\theta + i\sin\theta)$, and $r_2 = \rho(\cos\theta - i\sin\theta)$. Thus we may look for a solution for $a_n$ in the form:

$$\begin{aligned} a_n &= c_1 r_1^n + c_2 r_2^n \\ &= c_1\rho^n(\cos n\theta + i\sin n\theta) + c_2\rho^n(\cos n\theta - i\sin n\theta) \\ &= \rho^n((c_1+c_2)\cos n\theta + i(c_1-c_2)\sin n\theta) \\ &= \rho^n(K_1\cos n\theta + K_2\sin n\theta). \end{aligned}$$

Here $K_1 = (c_1+c_2)$ and $K_2 = i(c_1-c_2)$. In particular, it means that the expression $a_n = \rho^n(K_1\cos n\theta + K_2\sin n\theta)$ satisfies the recurrence relation we started with. We also notice that $K_1$ and $K_2$ are assumed to be real. Since $a_0$ and $a_1$ are given, we find them by substituting $n = 0$ and $n = 1$:

$$\left\{\begin{array}{lcl} K_1\cos 0 + K_2\sin 0 &=& a_0 \\ \rho(K_1\cos\theta + K_2\sin\theta) &=& a_1 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} K_1 &=& a_0 \\ \rho(K_1\cos\theta + K_2\sin\theta) &=& a_1 \end{array}\right.$$

We notice that the system always has a solution provided $\rho\sin\theta \neq 0$ for arbitrary initial coefficients $a_0$ and $a_1$. On the other hand, the condition $\rho\sin\theta = 0$ means that either $\rho = 0$ or $\sin\theta = 0$. Each of those imply that the roots $r_1$ $r_2$ are real.

We summarize the above discussion:

**Theorem 3.** Let $a_0$ and $a_1$ are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where $A, B$ are non-zero real constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two complex roots

$$r_1 = \rho(\cos\theta + i\sin\theta), \quad r_2 = \rho(\cos\theta - i\sin\theta),$$

Then $a_n = \rho^n(K_1\cos n\theta + K_2\sin n\theta)$, where the coefficients $K_1$, $K_2$ are determined by solving the system

$$\left\{\begin{array}{lcl} K_1 &=& a_0 \\ \rho(K_1\cos\theta + K_2\sin\theta) &=& a_1 \end{array}\right.$$

## The Method of Generating Functions.

There is another powerful technique to resolve recurrence relations. Let $a_0, a_1, \ldots, a_n, \ldots$ be a sequence of real numbers. Then the series

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

is called a *generating function* for the sequence $\{a_i\}$.

**Examples. (1)** The function

$$f(x) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

is a generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, \ldots$.

**(2)** We notice that $1 - x^{n+1} = (1-x)(1+x+x^2+\cdots+x^n)$. This gives the generating function

$$\frac{1-x^{n+1}}{1-x}$$

for the sequence $1, 1, \ldots, 1, 0, 0, \ldots$.

**(3)** Similarly to the previous example, we notice that $1 = (1-x)(1+x+x^2+\cdots+x^n+\cdots)$. This gives the generating function

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

for the sequence $1, 1, \ldots, 1, \ldots$.

**(4)** Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx}\frac{1}{1-x} = \sum_{i=0}^{\infty} \frac{d}{dx} x^i$$

Since $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$, we obtain the identity:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

Thus the function $\frac{1}{(1-x)^2}$ is a generating function for the sequence $1, 2, 3, 4, \ldots, n, \ldots$. We also notice that the function

$$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots$$

is a generating function for the sequence $0, 1, 2, 3, 4, \ldots, n, \ldots$.

**(5)** We take one more derivative: Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots)$$

Since $\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{x+1}{(1-x)^3}$, we obtain the identity:

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots + n^2x^{n-1} + \cdots$$

Thus the function $\frac{x+1}{(1-x)^3}$ is a generating function for the sequence $1^2, 2^2, 3^2, 4^2, \ldots, n^2, \ldots$. Then we see that the function

$$\frac{x(x+1)}{(1-x)^3} = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots + n^2x^n + \cdots$$

is a generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2, \ldots, n^2, \ldots$.