Summary on Lecture 2, January 6, 2015

Second Order Recurrence Relations

Example. Consider the sequence defined by $a_0 = 1$, $a_1 = -3$, and $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. The we try $a_n = cr^n$ with $c \neq 0$ to get the following characteristic equation: $r^2 - 6r + 9 = 0$. We obtain the solution $r = r_1 = r_2 = 3$. We notice that the $a_n = c_1r^n + c_2nr^n$ satisfies the relation $a_n = 6a_{n-1} - 9a_{n-2}$. We notice that $6 = 2r$ and $9 = r^2$. Then, indeed, we have:

$$\begin{aligned} c_1r^n + c_2nr^n &= 6c_1r^{n-1} + 6c_2(n-1)r^{n-1} - 9c_1r^{n-2} - 9c_2(n-2)r^{n-2} \\ &= c_1(6r^{n-1} - 9r^{n-2}) + c_2(2(n-1)r \cdot r^{n-1} - (n-2)r^2r^{n-2}) \\ &= c_1(6r^{n-1} - 9r^{n-2}) + c_2nr^n. \end{aligned}$$

This is true since $r^n = 6r^{n-1} - 9r^{n-2}$. Thus $a_n = c_1r^n + c_2nr^n = c_13^n + c_23^n$ satisfies the relation $a_n = 6a_{n-1} - 9a_{n-2}$. Then for $n = 0, 1$, we obtain:

$$\left\{ \begin{array}{rcl} 1 &=& c_1 \\ -3 &=& 3c_1 + 3c_2 \end{array} \right. \Longrightarrow \left\{ \begin{array}{rcl} 1 &=& c_1 \\ -1 &=& 1 + c_2 \end{array} \right. \Longrightarrow \left\{ \begin{array}{rcl} 1 &=& c_1 \\ -2 &=& c_2 \end{array} \right.$$

We obtain the answer $a_n = 3^n - 2n3^n$. This example is a particular case of the following Theorem:

Theorem 2. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has one real solution $r \neq 0$ (i.e., $r_1 = r_2 = r$) Then $a_n = c_1r^n + c_2nr^n$, where the constants c_1 and c_2 are determined by solving the system $\left\{ \begin{array}{rcl} a_0 &=& c_1 \\ a_1 &=& c_1r + c_2r \end{array} \right.$

Exercise: Prove Theorem 2.

Fibonacci numbers again: nontrivial application. Now we denote by F_n the Fibonnaci numbers defined above, i.e. $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let $\alpha = \frac{1+\sqrt{5}}{2}$. We need the following property:

Lemma 1. $F_n > \alpha^{n-2}$ for $n \geq 3$.

Exercise: Prove Lemma 1 by induction.

Let m, k be positive integers, $k \geq 2$, and we look at the division:

$$m = q \cdot k + r, \quad 0 \leq r < b.$$

Recall that a key to compute $\gcd(m, k)$ is the identity $\gcd(m, k) = \gcd(k, r)$. We organize the Euclidian Algorithm as follows to match the notations from the book.

Let $r_0 = m$, $r_1 = k$. Then we have the divisions:

$$\begin{array}{lcll} r_0 &=& q_1r_1 + r_2 & 0 \leq r_2 < r_1 \\ r_1 &=& q_2r_2 + r_3 & 0 \leq r_3 < r_2 \\ r_2 &=& q_3r_3 + r_4 & 0 \leq r_4 < r_3 \\ \cdots && \cdots & \cdots \\ r_{n-2} &=& q_{n-1}r_{n-1} + r_n & 0 \leq r_n < r_{n-1} \\ r_{n-1} &=& q_nr_n & \end{array} \tag{1}$$

Then we have the sequence of identities:

$$\gcd(m, k) = \gcd(r_0, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3) = \cdots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n.$$

We notice that we have performed n divisions, and every quotient $q_i \geq 1$ for all $i = 1, 2, \ldots, n-1$. Then the $r_{n-1} = q_nr_n$ and $r_n < r_{n-1}$ imply that $q_n \geq 2$.

Now we examine the remainders $r_n, r_{n-1}, \ldots, r_2, r_1$ (here $r_1 = k$). We have:

$$\begin{array}{llrcl}
r_n > 0, \text{ i.e. } r_n \geq 1 \text{ thus } r_n \geq F_2 = 1 & \text{i.e.} & r_n & \geq & F_2 \\
q_n \geq 2 \text{ and } r_n \geq 1 \text{ thus } r_{n-1} = q_n r_n \geq 2 \cdot 1 = 2 = F_3 & \text{i.e.} & r_{n-1} & \geq & F_3 \\
r_{n-2} = q_{n-1} r_{n-1} + r_n \geq 1 \cdot r_{n-1} + r_n \geq F_2 + F_3 = F_4 & \text{i.e.} & r_{n-2} & \geq & F_4 \\
r_{n-3} = q_{n-2} r_{n-2} + r_{n-1} \geq 1 \cdot r_{n-2} + r_{n-1} \geq F_3 + F_4 = F_5 & \text{i.e.} & r_{n-3} & \geq & F_5 \\
\ldots\ldots\ldots\ldots\ldots & \ldots\ldots & & & \\
r_2 = q_3 r_3 + r_4 \geq 1 \cdot r_3 + r_4 \geq F_{n-1} + F_{n-2} = F_n & \text{i.e.} & r_2 & \geq & F_n \\
r_1 = q_2 r_2 + r_3 \geq 1 \cdot r_2 + r_3 \geq F_n + F_{n-1} = F_{n+1} & \text{i.e.} & r_1 & \geq & F_{n+1}
\end{array}$$

Since $k = r_1$, we obtain $k \geq F_{n+1}$, $m \geq k \geq 2$. Lemma 1 then implies that

$$k \geq F_{n+1} \geq \alpha^{n+1-2} = \alpha^{n-1}, \quad \text{or} \quad \log_{10} k \geq (n-1) \log_{10} \alpha$$

Then we have that $\log_{10} \alpha = \log_{10}(\frac{1+\sqrt{5}}{2}) = 0.208... > 0.2 = \frac{1}{5}$, i.e., $\log_{10} k \geq \frac{n-1}{5}$. This means that if k is such that $10^{s-1} \leq k < 10^s$, then

$$s = \log_{10} 10^s > \log_{10} k \geq \frac{n-1}{5}, \quad \text{or} \quad n < 5s + 1.$$

We proved the following result.

Theorem 3. Let $m \geq k \geq 2$, and k has at most s digits, (i.e., $10^{s-1} \leq k < 10^s$). Then the Euclidian Algorithm requires at most $5s$ divisions to compute $\gcd(m, k)$.