

Summary on Lecture 5, April 9, 2018

The Method of Generating Functions.

There is another powerful technique to resolve recurrence relations. Let $a_0, a_1, \dots, a_n, \dots$ be a sequence of real numbers. Then the series

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

is called a *generating function* for the sequence $\{a_i\}$.

Examples. (1) The function

$$f(x) = (1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

is a generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}, 0, \dots$

(2) We notice that $1 - x^{n+1} = (1-x)(1+x+x^2+\dots+x^n)$. This gives the generating function

$$\frac{1-x^{n+1}}{1-x}$$

for the sequence $1, 1, \dots, 1, 0, 0, \dots$

(3) Similarly to the previous example, we notice that $1 = (1-x)(1+x+x^2+\dots+x^n+\dots)$. This gives the generating function

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

for the sequence $1, 1, \dots, 1, \dots$

(4) Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx} \frac{1}{1-x} = \sum_{i=0}^{\infty} \frac{d}{dx} x^i$$

Since $\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$, we obtain the identity:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1} + \dots = \sum_{i=0}^{\infty} (i+1)x^i.$$

Thus the function $\frac{1}{(1-x)^2}$ is a generating function for the sequence $1, 2, 3, 4, \dots, n, \dots$. We also notice that the function

$$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n + \dots = \sum_{i=0}^{\infty} ix^i$$

is a generating function for the sequence $0, 1, 2, 3, 4, \dots, n, \dots$

(5) We take one more derivative: Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx} \frac{x}{(1-x)^2} = \frac{d}{dx} (0 + x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n + \dots)$$

Since $\frac{d}{dx} \frac{x}{(1-x)^2} = \frac{x+1}{(1-x)^3}$, we obtain the identity:

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots + n^2x^{n-1} + \dots = \sum_{i=0}^{\infty} (i+1)^2x^i$$

Thus the function $\frac{x+1}{(1-x)^3}$ is a generating function for the sequence $1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots$. Then we see that the function

$$\frac{x(x+1)}{(1-x)^3} = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots + n^2x^n + \dots = \sum_{i=0}^{\infty} i^2x^i$$

is a generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots$.

Before analyzing the examples, we need one more identity on generation function:

$$\frac{1}{(1+x)^k} = \sum_{n=0}^{\infty} (-1)^n \binom{k+n-1}{n} x^n \quad (1)$$

This could be verified using Taylor-Maclaurin decomposition of the function $\frac{1}{(1+x)^k}$.

Example 1. Let $a_0 = 1$, and $a_n - 3a_{n-1} = n$, $n \geq 1$. This is new type of recurrence relations: *non-homogeneous*. We write the first few terms:

$$\begin{array}{rcl} a_1 - 3a_0 & = & 1 \\ a_2 - 3a_1 & = & 2 \\ a_3 - 3a_2 & = & 3 \\ \dots & & \dots \\ a_n - 3a_{n-1} & = & n \\ \dots & & \dots \end{array}$$

We multiply the first equation by x , the second, by x^2 , the third, by x^3 and so on. We get:

$$\begin{array}{rcl} a_1x - 3a_0x & = & x \\ a_2x^2 - 3a_1x^2 & = & x^2 \\ a_3x^3 - 3a_2x^3 & = & 3x^3 \\ \dots & & \dots \\ a_nx^n - 3a_{n-1}x^n & = & nx^n \\ \dots & & \dots \end{array}$$

We add the terms of the equations to get the identity:

$$\sum_{n=1}^{\infty} a_nx^n - 3x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \sum_{n=1}^{\infty} nx^n. \quad (2)$$

We denote $f(x) = \sum_{n=0}^{\infty} a_nx^n$, then

$$\sum_{n=1}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^n - a_0 = f(x) - 1,$$

$$\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = f(x).$$

Also, we recall from Example 4, we have

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \dots + nx^n + \dots = \frac{x}{(1-x)^2}$$

Then the identity (2) becomes

$$(f(x) - 1) - 3xf(x) = \frac{x}{(1-x)^2} \quad (3)$$

Then the identity (3) becomes

$$\begin{aligned}
(f(x) - 1) - 3xf(x) &= -1 + f(x)(1 - 3x) = \frac{x}{(1-x)^2}, \quad \text{or} \\
f(x)(1 - 3x) &= \frac{x}{(1-x)^2} + 1, \quad \text{or} \\
f(x) &= \frac{x}{(1-x)^2(1-3x)} + \frac{1}{(1-3x)}
\end{aligned} \tag{4}$$

Now we would like to decompose the term $\frac{x}{(1-x)^2(1-3x)}$ as the sum using so called method of partial fractions:

$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-3x)}$$

We obtain:

$$\begin{aligned}
x &= A(1-x)(1-3x) + B(1-3x) + C(1-x)^2 \\
&= 3Ax^2 - 4Ax + A + B - 3Bx + C - 2Cx + Cx^2 \\
&= (3A + C)x^2 + (-4A - 3B - 2C)x + (A + B + C)
\end{aligned}$$

We get the system:

$$\begin{cases} 3A + C = 0 \\ -4A - 3B - 2C = 1 \\ A + B + C = 0 \end{cases} \quad \text{or} \quad \begin{cases} C = -3A \\ -4A - 3B + 6A = 1 \\ A + B - 3A = 0 \end{cases} \quad \text{or} \quad \begin{cases} C = -3A \\ 2A - 3B = 1 \\ -2A + B = 0 \end{cases}$$

The equations

$$\begin{cases} 2A - 3B = 1 \\ -2A + B = 0 \end{cases}$$

give $-2B = 1$, i.e. $B = -\frac{1}{2}$, and $A = \frac{1}{2}B = -\frac{1}{4}$. Then the equation $C = -3A$ gives $C = \frac{3}{4}$. We obtain:

$$\begin{aligned}
f(x) &= -\frac{1}{4} \cdot \frac{1}{(1-x)} - \frac{1}{2} \cdot \frac{1}{(1-x)^2} + \frac{3}{4} \cdot \frac{1}{(1-3x)} + \frac{1}{(1-3x)} \\
&= -\frac{1}{4} \cdot \frac{1}{(1-x)} - \frac{1}{2} \cdot \frac{1}{(1-x)^2} + \frac{7}{4} \cdot \frac{1}{(1-3x)}
\end{aligned}$$

Then we have

$$\begin{aligned}
-\frac{1}{4} \cdot \frac{1}{(1-x)} &= -\frac{1}{4}(1 + x + x^2 + x^3 + \dots + x^n + \dots) = \sum_{n=0}^{\infty} \left(-\frac{1}{4}x^n\right) \\
-\frac{1}{2} \cdot \frac{1}{(1-x)^2} &= -\frac{1}{2} \sum_{n=0}^{\infty} \left((-1)^n \binom{2+n-1}{n} (-x)^n\right) = \sum_{n=0}^{\infty} \left(-\frac{1}{2} \binom{2+n-1}{n} x^n\right) \\
\frac{7}{4} \cdot \frac{1}{(1-3x)} &= \frac{7}{4} \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} \left(\frac{7 \cdot 3^n}{4} x^n\right)
\end{aligned}$$

We notice that $\binom{2+n-1}{n} = \binom{n+1}{n} = \binom{n+1}{1} = (n+1)$. Then we add all three series together to get

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{4} - \frac{n+1}{2} + \frac{7 \cdot 3^n}{4}\right) x^n = \sum_{n=0}^{\infty} \left(-\frac{3}{4} - \frac{n}{2} + \frac{7 \cdot 3^n}{4}\right) x^n.$$

We obtain the answer: $a_n = -\frac{3}{4} - \frac{n}{2} + \frac{7 \cdot 3^n}{4} = \frac{7 \cdot 3^n - 3}{4} - \frac{n}{2}$.

Remark. We notice that for $n \geq 1$, the coefficient $\frac{3 \cdot 7(3^{n-1} - 1)}{4} - \frac{n}{2}$ is always an integer. Prove it!