Summary on Lecture 2, April 3, 2018

Recurrence Relations

Let $a_n = Aa_{n-1} + Ba_{n-2}$ be a second order recurrence relation. Then the equation $r^2 - Ar - B = 0$ is called a *characteristic equation* of that relation.

We are ready to prove the following result:

Theorem 1. Let a_0 and a_1 be given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a second order recurrence relation, $n \geq 2$, where A, B are non-zero constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two real different real solutions r_1 and r_2. Then $a_n = c_1 r_1^n + c_2 r_2^n$, where the constants c_1 and c_2 are determined by solving the system

$$\begin{cases} a_0 &= c_1 + c_2 \\ a_1 &= c_1 r_1 + c_2 r_2 \end{cases}$$

Proof. Indeed, we look for a solution $a_n = cr^n$, then the recurrence realtion $a_n = Aa_{n-1} + Ba_{n-2}$ gives the characteristic equation $r^2 - Ar - B = 0$. By assumption, there are two two different real solutions, r_1 and r_2 of $r^2 - Ar - B = 0$. Then the sum $c_1 r_1^n + c_2 r_2^n$ will satisfy the recurrence. Finally, we notice that the system $\begin{cases} a_0 &= c_1 + c_2 \\ a_1 &= c_1 r_1 + c_2 r_2 \end{cases}$ always have a unique solution if $r_1 \neq r_2$ (Explain why).

Next question: How to solve this problem if $r_1 = r_2$?

Example. Consider the sequence defined by $a_0 = 1$, $a_1 = -3$, and $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. The we try $a_n = cr^n$ with $c \neq 0$ to get the following characteristic equation: $r^2 - 6r + 9 = 0$. We obtain the solution $r = r_1 = r_2 = 3$. We notice that the $a_n = c_1 r^n + c_2 n r^n$ satisfies the relation $a_n = 6a_{n-1} - 9a_{n-2}$. We notice that $6 = 2r$ and $9 = r^2$. Then, indeed, we have:

$$\begin{aligned} c_1 r^n + c_2 n r^n &= 6c_1 r^{n-1} + 6c_2(n-1)r^{n-1} - 9c_1 r^{n-2} - 9c_2(n-2)r^{n-2} \\ &= c_1(6r^{n-1} - 9r^{n-2}) + c_2(2(n-1)r \cdot r^{n-1} - (n-2)r^2 r^{n-2}) \\ &= c_1(6r^{n-1} - 9r^{n-2}) + c_2 n r^n. \end{aligned}$$

This is true since $r^n = 6r^{n-1} - 9r^{n-2}$. Thus $a_n = c_1 r^n + c_2 n r^n = c_1 3^n + c_2 3^n$ satisfies the relation $a_n = 6a_{n-1} - 9a_{n-2}$. Then for $n = 0, 1$, we obtain:

$$\begin{cases} 1 &= c_1 \\ -3 &= 3c_1 + 3c_2 \end{cases} \implies \begin{cases} 1 &= c_1 \\ -1 &= 1 + c_2 \end{cases} \implies \begin{cases} 1 &= c_1 \\ -2 &= c_2 \end{cases}$$

We obtain the answer $a_n = 3^n - 2n3^n$. This example is a particular case of the following Theorem:

Theorem 2. Let a_0 and a_1 be given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has one real solution $r \neq 0$ (i.e., $r_1 = r_2 = r$) Then $a_n = c_1 r^n + c_2 n r^n$, where the constants c_1 and c_2 are determined by solving the system $\begin{cases} a_0 &= c_1 \\ a_1 &= c_1 r + c_2 r \end{cases}$

Exercise: Prove Theorem 2.

Fibonacci numbers again: nontrivial application. Now we denote by F_n the Fibonnaci numbers defined above, i.e. $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let $\alpha = \frac{1+\sqrt{5}}{2}$. We need the following property:

Lemma 1. $F_n > \alpha^{n-2}$ for $n \geq 3$.

Exercise: Prove Lemma 1 by induction.