Summary on Lecture 14, May 4, 2018

Trees: definitions and basic properties

Definition. A connected graph $G=(V,E)$ is a *tree* if G has no cycles.

Theorem 1. *Let $T=(V,E)$ be a tree, and $v,v'\in V(T)$. Then there exists a unique path from v to v'.*

Proof. Assume there are two different paths connecting v and v'. Then there is a cycle. □

Definition. Let $G=(V,E)$ be a graph. A tree $T\subset G$ is a *spanning tree of* G if $V(T)=V(G)$.

Theorem 2. *Let $G=(V,E)$ be a finite graph. Then G is connected if and only if there exists a spanning tree T of G.*

Proof. Let $T\subset G$ be a spanning tree. Then for any two vertices $v,v'\in V(T)=V(G)$ there exists a path in T (and, consequently, in G) which connects v and v'.

On the other hand, if G is connected, then we remove all loops from G to obtain graph G_1, which is still connected. If G_1 is a tree, we are done. If not, we find a cycle in G_1, and choose an edge e_1 in that cycle. Then $G_2=G_1\setminus\{e_1\}$ is a connected graph. If G_2 has a cycle, we repeat the procedure. Finally we will obtain a connected graph $G_k\subset G$ which has no cycles since the original graph is finite, i.e., G_k is a spanning tree. □

Let $G=(V,E)$ be a graph. We say that a vertex $v\in V$ is a *leaf* of G if $\deg v=1$. We need the following observation:

Lemma 1. *Let $T=(V,E)$ be a finite tree. There there are at least two leaves v,v' in T.*

Proof. Consider a longest path in T, and let v and v' are its end-points. We notice that $v\neq v'$ since T is a tree. Then v and v' are both leaves, indeed, if not, we can extend a path by at least one edge. □

Theorem 2. *Let $T=(V,E)$ be a finite tree. Then $|V|=|E|+1$.*

Proof. Induction on $k=|V|$. Theorem 2 obviously holds if $k=1$ and $k=2$. Assume Theorem 2 holds for all trees $T'=(V',E')$ with $|V|\leq n$. Consider a tree $T=(V,E)$ with $|V|=n+1$. Then by Lemma 1, there exists a leaf $v\in V'$ with a single edge e. We prune the tree T at v, to obtain a tree $T'=(V',E')$, where $V'=V\setminus\{v\}$, $E'=E\setminus\{e\}$. By induction, $|V'|=|E'|+1$. Since $|V|=|V'|+1$ and $|E|=|E'|+1$, we obtain that $|V|=|E|+1$. □

Theorem 3. *The following statements are equivalent:*

(a) *A graph $G=(V,E)$ is a finite tree.*

(b) *A graph $G=(V,E)$ is connected, but a removal of any edge will make it disconnected.*

(c) *A graph $G=(V,E)$ contains no cycles and $|V|=|E|+1$.*

(d) *A graph $G=(V,E)$ is connected and $|V|=|E|+1$.*

Proof. (a)$\Longrightarrow$(b) Let $G=(V,E)$ be a finite tree. Assume that a removing an edge e from G keeps $G\setminus\{e\}$ connected. Let v,v' be the end-vertices of e. Then there exists a path in $G\setminus\{e\}$ connecting v and v'. Then we put the edge e back an we obtain a cycle. Thus G could not be a tree in the first place. Contradiction. Thus (a)$\Longrightarrow$(b).

(b)$\Longrightarrow$(c) Let $G=(V,E)$ be as in (b), however, G contains a cycle. Then we can remove an edge e from such a cycle, and the graph $G\setminus\{e\}$ is still connected. Contradiction. Thus $G=(V,E)$ has no cycles, and, by definition, G is a tree. By Theorem 2, $|V|=|E|+1$.

(c)$\Longrightarrow$(d) Let $G=(V,E)$ be as in (c), however, G is not connected. It means that $G=G_1\cup\cdots\cup G_r$, where G_i are connective components of G and $r\geq 2$. Then every component G_i has no cycles and it is connected, thus G_i is a tree by definition. Theorem 2 gives that

$$|V(G_1)| = |E(G_1)| + 1, \ \cdots \ |V(G_r)| = |E(G_r)| + 1.$$

We obtain:

$$|V(G)| = |V(G_1)| + \cdots + |V(G_r)| = |E(G_1)| + \cdots + |E(G_r)| + r = |E(G)| + r.$$

At the same time, we have that $|V(G)| = |E(G)| + 1$. Thus $r = 1$. Contradiction.

(d)$\Longrightarrow$(a) Let $G = (V, E)$ be as in (d). Then G has a spanning tree $T \subset G$ by Theorem 1. Then $V(T) = V(G)$ and $|E(T)| \leq |E(G)|$. Since T is a tree, $|V(T)| = |E(T)| + 1$, and by assumption, $|V(G)| = |E(G)| + 1$. Thus $|E(T)| = |E(G)|$, i.e., $T = G$. In particular, G is a tree (in fact, it is its own spanning tree). $\square$