Summary on Lecture 13, May 2, 2018

More on Chromatic Polynomials

Let $G = (V, E)$ be a graph, and e be its edge with vertices a and b. We denote by G_e the graph which is obtained by removing the edge e. Let G'_e be a graph which is obtained from G_e by identifying the vertices a and b, see Fig. 3. Last time we proved the following Theorem:

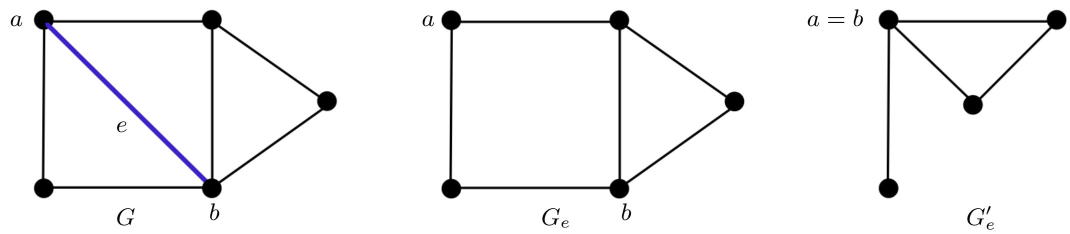

Fig. 3. The graphs G, G_e and G'_e

Theorem 1. *Let $G = (V, E)$ be a connected graph, and $e \in E$. Then*

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda).$$

Proof. Let $e = \{a, b\}$. Consider the value $P(G_e, \lambda)$. There are two possibilities here: either the vertices a and b have the same color or not. If they are of different colors, then it corresponds to a proper coloring of G. If they are the same, then it corresponds to a proper coloring of G'_e. □

Lemma 1. *Let T be a tree with n vertices. Then $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$.*

Proof. Induction on n. If $n = 1$, then obviously $P(T, \lambda) = \lambda$. Let $n > 1$. We find an edge e such that $e = \{a, b\}$, where a is a leaf. Then T_e is a disjoint union of a tree on $(n - 1)$ vertices and a single vertex, and T'_e is a tree on $(n - 1)$ vertices, see Fig. 4

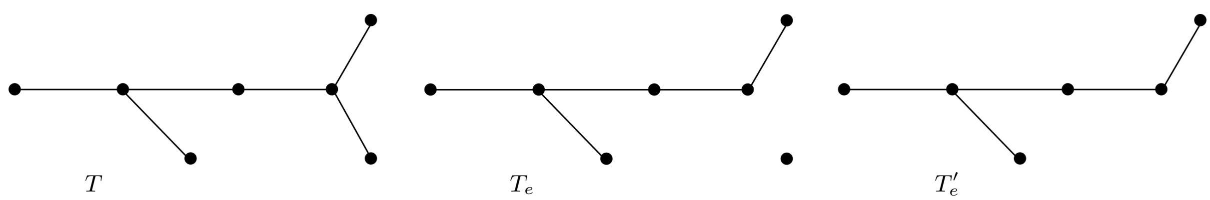

Fig. 4. The graphs T, T_e and T'_e

By induction, we have that $P(T'_e, \lambda) = \lambda(\lambda - 1)^{n-2}$, and $P(T_e, \lambda,) = \lambda(\lambda - 1)^{n-2} \cdot \lambda = \lambda^2(\lambda - 1)^{n-2}$. Then

$$P(T, \lambda) = P(T_e, \lambda) - P(T'_e, \lambda) = \lambda^2(\lambda - 1)^{n-2} - \lambda(\lambda - 1)^{n-2} = \lambda(\lambda - 1)^{n-1}.$$

This completes the induction. □

Lemma 2. *Let C_n be a cycle on n vertices. Then $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.*

Proof. Induction on n. If $n = 3$, $C_3 = K_3$, so we check

$$\begin{aligned} P(K_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2) &= (\lambda - 1)^3 + (-1)^3(\lambda - 1) = (\lambda - 1)((\lambda - 1)^2 - 1) \\ &= (\lambda - 1)(1 - 2\lambda + \lambda^2 - 1) \\ &= \lambda(\lambda - 1)(\lambda - 2). \end{aligned}$$

Let $n > 3$, and e be an edge of C_n. Then $(C_n)_e$ is just a path on n vertices, and $(C_n)'_e = C_{n-1}$ is just a cycle on $(n-1)$ vertices. We have

$$\begin{aligned} P(C_n, \lambda) &= P((C_n)_e, \lambda) - P(C_{n-1}, \lambda) \\ &= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} - (-1)^{n-1}(\lambda-1) \\ &= (\lambda-1)^n + (-1)^n(\lambda-1). \end{aligned}$$

This completes the induction. □

Remark. We notice that $P(C_n, 1) = 0$, and $P(C_n, 2) = 1 + (-1)^n$. Hence $P(C_n, 2) = 2$ if n is even, and $P(C_n, 2) = 0$ if n is odd. Let $n = 2k+1$, then $P(C_{2k+1}, 3) = 2^{2k+1} - 2 = 2(2^{2k} - 1)$. We conclude that $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if $n = 2k+1$.

We define a *wheel on* $(n+1)$ *vertices* W_{n+1} by taking a cycle on n vertices and connecting each vertex of a cycle to one more $(n+1)$the vertex.

Lemma 3. *Let* W_{n+1} *be a wheel on* $(n+1)$ *vertices. Then* $P(W_{n+1}, \lambda) = \lambda(\lambda-2)^n - (-1)^n\lambda(\lambda-2)$.

Proof. We assign an arbitrary color to the centeral vertex, then we can use $(\lambda - 1)$ colors for the remaining vertices. We obtain:

$$P(W_{n+1}, \lambda) = \lambda\left[(\lambda-1-1)^n - (-1)^n(\lambda-1-1)\right] = \lambda(\lambda-2)^n - (-1)^n\lambda(\lambda-2).$$

This proves Lemma 3. □

We make two observations about the chromatic polynomial. Let $G = (V, E)$, and

$$P(G, \lambda) = a_0 + a_1\lambda + \cdots + a_d\lambda^d.$$

(1) The coefficient $a_0 = 0$. Indeed, $P(G, 0) = a_0$ and at the same time is a number of proper colorings of G with 0 colors, i.e., $a_0 = 0$.

(2) Assume that $|E| > 0$. Then $a_1 + a_2 + \cdots + a_d = P(G, 1) = 0$. Indeed, if G contains an edge, then we cannot give proper colorings of G with one color.

Lemma 4. *Let* G_1 *and* G_2 *share a single vertex, i.e.,* $V(G_1) \cap V(G_2) = \{v\}$*. Then*

$$P(G_1 \cup G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda}.$$

Proof. Let us give a coloring to G_2. Then the vertex v receives some color λ_0. Then we count how many ways to color G_1 so that v is colored with given color λ_0. Let $Q(G_1; v, \lambda, \lambda_0)$ be the result. Because of the symmetry with respect to colors, $Q(G_1; v, \lambda, \lambda_0)$ is the same for any choice of λ_0. On the other hand, by taking every value of λ_0, we count all colorings of the graph G_1. We conclude that $Q(G_1; v, \lambda, \lambda_0)\lambda = P(G_1, \lambda)$, or

$$Q(G_1; v, \lambda, \lambda_0) = \frac{P(G_1, \lambda)}{\lambda}$$

We obtain that all colorings of $G_1 \cup G_2$ are given by

$$\frac{P(G_1, \lambda)}{\lambda} \cdot P(G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda}.$$

This completes the proof. □

Lemma 5. *Let* G_1 *and* G_2 *be such that* $G_1 \cap G_2 = K_n$*. Then*

$$P(G_1 \cup G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{P(K_n, \lambda)} = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}.$$

(Recall that $\lambda^{(n)} := \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$.)

Exercise. Prove Lemma 5.