Summary on Lecture 12, April 30, 2018

Graph Coloring and Chromatic Polynomials

Graph coloring, or more specifically vertex coloring means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

This definition allows us to use a separate color for each vertex. From a mathematical perspective graph coloring is only interesting if we restrict the permissible colors to a fixed finite set S. It is easy to see that the choice of actual colors is irrelevant, and therefore any graph property related to coloring may only depend on the cardinality $|S| = k$. We may as well label the nodes using the numbers $1, 2, \ldots, k$.

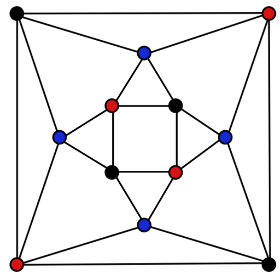

Fig. 1. Graph G_1 with $\chi(G_1) = 3$

Definition. Let $G = (V, E)$ be a graph. A *proper λ-coloring of a graph G* is a function $\sigma : V \to \{1, 2, \ldots, \lambda\}$ which satisfies $\sigma(v) \neq \sigma(v')$ for any edge $e = \{v, v'\}$. Note that it is not compulsory to use all the colors. The graph is said to be *λ-colorable* if such a function exists. The *chromatic number* $\chi(G)$ is the minimal λ for which the graph G is λ-colorable.

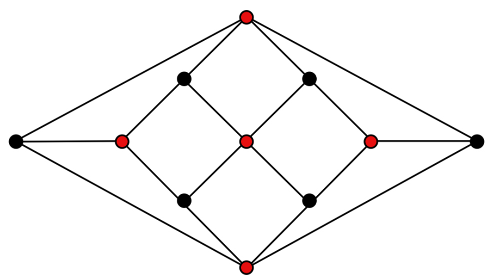

Fig. 2. Graph G_2 with $\chi(G_2) = 2$

Example. It is easy to see that $\chi(K_n) = n$.

In order to determine the chromatic number $\chi(G)$, we consider more general problem of finding the chromatic polynomial $P(G, \lambda)$.

Definition. For each $\lambda = 0, 1, 2, \ldots$ the value of the chromatic polynomial $P(G, \lambda)$ is the number of different proper λ-colorings of G. Here we identify colorings $\sigma : V \to \{1, 2, \ldots, \lambda\}$ and $\sigma' : V \to \{1, 2, \ldots, \lambda\}$ if $\sigma = \sigma'$ as functions.

Examples. (1) Let $G = (V, E)$ with $|V| = n$ and $E = \emptyset$. Then $P(G, \lambda) = \lambda^n$.

(2) $P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$. We use the notation $\lambda^{(n)} := \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$.

(3) Let C be a path with n vertices. Then $P(C, \lambda) = \lambda(\lambda - 1)^{n-1}$. Similarly, $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$ for any tree with n vertices.

Let $G = (V, E)$ be a graph, and e be its edge with vertices a and b. We denote by G_e the graph which is obtained by removing the edge e. Let G'_e be a graph which is obtained from G_e by identifying the vertices a and b, see Fig. 3.

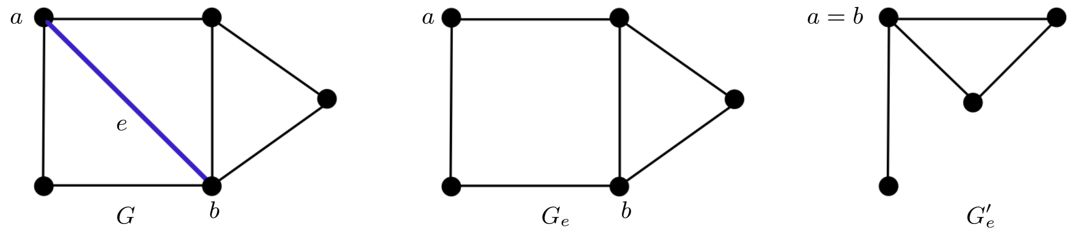

Fig. 3. The graphs G, G_e and G'_e

Theorem 1. *Let $G = (V, E)$ be a connected graph, and $e \in E$. Then*

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda).$$

Proof. Let $e = \{a, b\}$. Consider the value $P(G_e, \lambda)$. There are two possibilities here: either the vertices a and b have the same color or not. If they are of different colors, then it corresponds to a proper coloring of G. If they are the same, then it corresponds to a proper coloring of G'_e. □

Lemma 1. *Let T be a tree with n vertices. Then $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$.*

Proof. Induction on n. If $n = 1$, then obviously $P(T, \lambda) = \lambda$. Let $n > 1$. We find an edge e such that $e = \{a, b\}$, where a is a leaf. Then T_e is a disjoint union of a tree on $(n-1)$ vertices and a single vertex, and T'_e is a tree on $(n-1)$ vertices, see Fig. 4

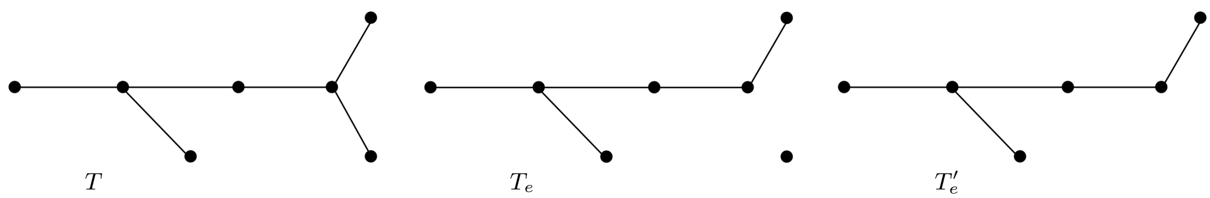

Fig. 4. The graphs T, T_e and T'_e

By induction, we have that $P(T'_e, \lambda) = \lambda(\lambda - 1)^{n-2}$, and $P(T_e, \lambda) = \lambda(\lambda - 1)^{n-2} \cdot \lambda = \lambda^2(\lambda - 1)^{n-2}$. Then

$$P(T, \lambda) = P(T_e, \lambda) - P(T'_e, \lambda) = \lambda^2(\lambda - 1)^{n-2} - \lambda(\lambda - 1)^{n-2} = \lambda(\lambda - 1)^{n-1}.$$

This completes the induction. □