

Summary on Lecture 7, July 28, 2015

Graph Coloring and Chromatic Polynomials

Graph coloring, or more specifically vertex coloring means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

This definition allows us to use a separate color for each vertex. From a mathematical perspective graph coloring is only interesting if we restrict the permissible colors to a fixed finite set S . It is easy to see that the choice of actual colors is irrelevant, and therefore any graph property related to coloring may only depend on the cardinality $|S| = k$. We may as well label the nodes using the numbers $1, 2, \dots, k$.

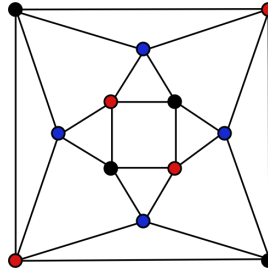


Fig. 1. Graph G_1 with $\chi(G_1) = 3$

Definition. Let $G = (V, E)$ be a graph. A *proper λ -coloring* of a graph G is a function $\sigma : V \rightarrow \{1, 2, \dots, \lambda\}$ which satisfies $\sigma(v) \neq \sigma(v')$ for any edge $e = \{v, v'\}$. Note that it is not compulsory to use all the colors. The graph is said to be *λ -colorable* if such a function exists. The *chromatic number* $\chi(G)$ is the minimal λ for which the graph G is λ -colorable.

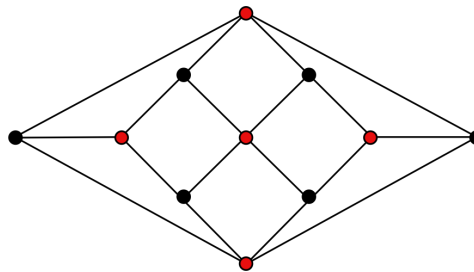


Fig. 2. Graph G_2 with $\chi(G_2) = 3$

Example. It is easy to see that $\chi(K_n) = n$.

In order to determine the chromatic number $\chi(G)$, we consider more general problem of finding the chromatic polynomial $P(\lambda, G)$.

Definition. For each $\lambda = 0, 1, 2, \dots$ the value of the chromatic polynomial $P(\lambda, G)$ is the number of different proper λ -colorings of G . Here we identify colorings $\sigma : V \rightarrow \{1, 2, \dots, \lambda\}$ and $\sigma' : V \rightarrow \{1, 2, \dots, \lambda\}$ if $\sigma = \sigma'$ as functions.

Examples. (1) Let $G = (V, E)$ with $|V| = n$ and $E = \emptyset$. Then $P(\lambda, G) = \lambda^n$.

(2) $P(K_n, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$. We use the notation $\lambda^{(n)} := \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$.

(3) Let C be a path with n vertices. Then $P(C, \lambda) = \lambda(\lambda - 1)^{n-1}$. Similarly, $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$ for any tree with n vertices.

Let $G = (V, E)$ be a graph, and e be its edge with vertices a and b . We denote by G_e the graph which is obtained by removing the edge e . Let G'_e be a graph which is obtained from G_e by identifying the vertices a and b , see Fig. 3.

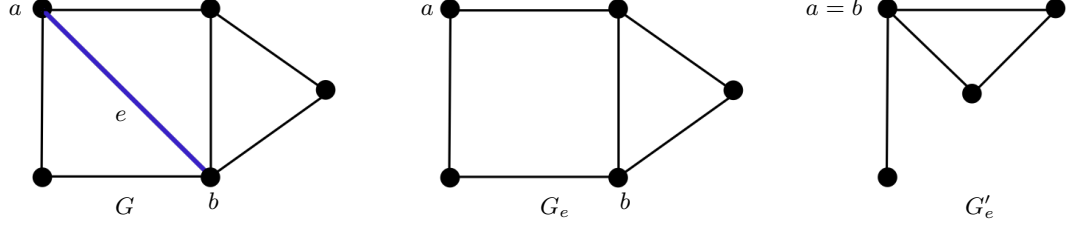


Fig. 3. The graphs G , G_e and G'_e

Theorem 1. Let $G = (V, E)$ be a connected graph, and $e \in E$. Then

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda).$$

Proof. Let $e = \{a, b\}$. Consider the value $P(G_e, \lambda)$. There are two possibilities here: either the vertices a and b have the same color or not. If they are of different colors, then it corresponds to a proper coloring of G . If they are the same, then it corresponds to a proper coloring of G'_e . \square

Lemma 1. Let T be a tree with n vertices. Then $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$.

Proof. Induction on n . If $n = 1$, then obviously $P(T, \lambda) = \lambda$. Let $n > 1$. We find an edge e such that $e = \{a, b\}$, where a is a leaf. Then T_e is a disjoint union of a tree on $(n - 1)$ vertices and a single vertex, and T'_e is a tree on $(n - 1)$ vertices, see Fig. 4

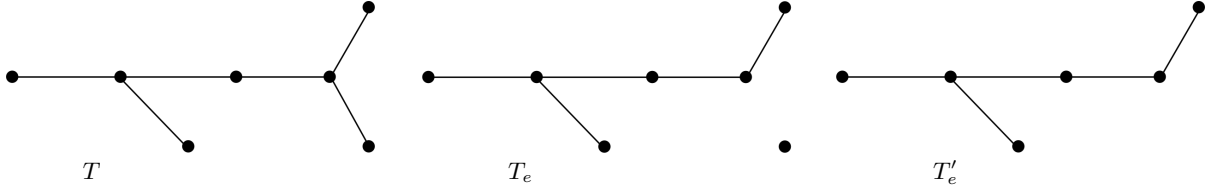


Fig. 4. The graphs T , T_e and T'_e

By induction, we have that $P(\lambda, T'_e) = \lambda(\lambda - 1)^{n-2}$, and $P(\lambda, T_e) = \lambda(\lambda - 1)^{n-2} \cdot \lambda = \lambda^2(\lambda - 1)^{n-2}$. Then

$$P(\lambda, T) = P(\lambda, T_e) - P(\lambda, T'_e) = \lambda^2(\lambda - 1)^{n-2} - \lambda(\lambda - 1)^{n-2} = \lambda(\lambda - 1)^{n-1}.$$

This completes the induction. \square

Lemma 2. Let C_n be a cycle on n vertices. Then $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.

Proof. Induction on n . If $n = 3$, $C_3 = K_3$, so we check

$$P(K_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2) = (\lambda - 1)^2 + (-1)^2(\lambda - 1) = (\lambda - 1)(\lambda - 1 - 1).$$

Let $n > 3$, and e be an edge of C_n . Then $(C_n)_e$ is just a path on n vertices, and $(C_n)'_e = C_{n-1}$ is just a cycle on $(n - 1)$ vertices. We have

$$\begin{aligned} P(C_n, \lambda) &= P((C_n)_e, \lambda) - P(C_{n-1}, \lambda) \\ &= \lambda(\lambda - 1)^{n-1} - (\lambda - 1)^{n-1} - (-1)^{n-1}(\lambda - 1) \\ &= (\lambda - 1)^n + (-1)^n(\lambda - 1). \end{aligned}$$

This completes the induction. \square

Remark. We notice that $P(C_n, 1) = 0$, and $P(C_n, 2) = 1 + (-1)^n$. Hence $P(C_n, 2) = 2$ if n is even, and $P(C_n, 2) = 0$ if n is odd. Let $n = 2k + 1$, then $P(C_{2k+1}, 3) = 2^{2k+1} - 2 = 2(2^{2k} - 1)$. We conclude that $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 2(2^{2k} - 1)$ if $n = 2k + 1$.

We define a *wheel* on $(n + 1)$ vertices W_{n+1} by taking a cycle on n vertices and connecting each vertex of a cycle to one more $(n + 1)$ the vertex.

Lemma 3. *Let W_{n+1} be a wheel on $(n + 1)$ vertices. Then $P(W_{n+1}, \lambda) = \lambda(\lambda - 2)^n - (-1)^n\lambda(\lambda - 2)$.*

Proof. We assign an arbitrary color to the central vertex, then we can use $(\lambda - 1)$ colors for the remaining vertices. We obtain:

$$P(W_{n+1}, \lambda) = \lambda[(\lambda - 1 - 1)^n - (-1)^n(\lambda - 1 - 1)] = \lambda(\lambda - 2)^n - (-1)^n\lambda(\lambda - 2).$$

This proves Lemma 3. □

We make two observations about the chromatic polynomial. Let $G = (V, E)$, and

$$P(G, \lambda) = a_0 + a_1\lambda + \cdots + a_d\lambda^d.$$

- (1) The coefficient $a_0 = 0$. Indeed, $P(G, 0) = a_0$ and at the same time is a number of proper colorings of G with 0 colors, i.e., $a_0 = 0$.
- (2) Assume that $|E| > 0$. Then $a_1 + a_2 + \cdots + a_d = P(G, 1) = 0$. Indeed, if G contains an edge, then we cannot give proper colorings of G with 1 color.

Lemma 4. *Let G_1 and G_2 share a single vertex, i.e., $V(G_1) \cap V(G_2) = \{v\}$. Then*

$$P(G_1 \cup G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda}.$$

Proof. Let us give a coloring to G_2 . Then the vertex v receives some color λ_0 . Then we count how many ways to color G_1 so that v is colored with given color λ_0 . Let $Q(G_1; v, \lambda, \lambda_0)$ be the result. Because of the symmetry with respect to colors, $Q(G_1; v, \lambda, \lambda_0)$ is the same for any choice of λ_0 . On the other hand, by taking every value of λ_0 , we count all colorings of the graph G_1 . We conclude that $Q(G_1; v, \lambda, \lambda_0)\lambda = P(G_1, \lambda)$, or

$$Q(G_1; v, \lambda, \lambda_0) = \frac{P(G_1, \lambda)}{\lambda}$$

We obtain that all colorings of $G_1 \cup G_2$ are given by

$$\frac{P(G_1, \lambda)}{\lambda} \cdot P(G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda}.$$

This completes the proof. □

Lemma 5. *Let G_1 and G_2 be such that $G_1 \cap G_2 = K_n$. Then*

$$P(G_1 \cup G_2, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{P(K_n, \lambda)} = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}.$$

(Recall that $\lambda^{(n)} := \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$.)

Exercise. Prove Lemma 5.