

Summary on Lecture 4, July 23, 2015

Introduction to Graph Theory. Definitions and Examples.

A graph G is given by three objects: a set $V = V(G)$ of vertices, a set $E = E(G)$ of edges and an assignment of the end vertices to every edge.

We will distinguish graphs and directed graphs (or digraphs), where each edge has a direction and its two vertices could be thought as its beginning and its end.

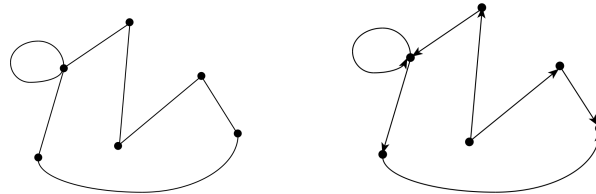


Fig. 1. Graph and digraph

Definition 1. Let V be an non-empty set, and $E \subset V \times V$. Then the pair $G = (V, E)$ is a *directed graph*. If

$$E \subset \{ \{v_0, v_1\} \mid v_0, v_1 \in V \}$$

(where $\{ \{v_0, v_1\} \mid v_0, v_1 \in V \}$ is the set of two-element subsets of V), then the pair $G = (V, E)$ is a *graph*, see Fig. 1. The set $V = V(G)$ is a set of vertices, and $E = E(G)$ is a set of edges of G .

It is convenient to denote an edge e as a pair of verices $e = \{v, v'\}$ (for a graph) and as an ordered pair $e = (v, v')$ (for a digraph). A loop is an edge e with the same vertices $v = v'$. Below we assume that a graph (or digraph) G has no loops.

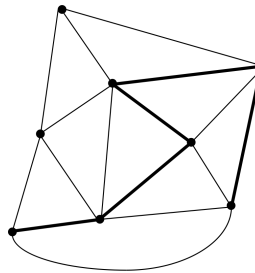


Fig. 2. A walk in G

Let $G = (V, E)$ be a graph, and $x, x' \in V$ be two vertices. An x - x' -walk is a finite alternatig sequence

$$x = x_0, e_1, x_1, e_2, x_2, \dots, x_{n-1}, e_n, x_n = x'$$

of vertices and edges, where $e_i = \{x_{i-1}, x_i\}$ for $i = 1, 2, \dots, n$, see Fig. 2. If $x = x'$, then an x - x' -walk is a *closed walk*. Otherwise, the walk is *open*. There are special types of walks:

- If no edge is repeated, then an x - x' -walk is an x - x' -*trail*.
- A closed trail is called a *circuit*.
- If no vertex is repeated, the an x - x' -walk is called an x - x' -*path*.
- If $x = x'$, then an x - x' -path is called a *cycle*.

Theorem 1. Let $G = (V, E)$ be a graph with $x, x' \in V$, $x \neq x'$. If there exists an x - x' -trail, then there exists an x - x' -path.

Proof. Since there exists a trail from x to x' , then there exists a trail of a shortest length. Let

$$\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{n-1}, x_n\}$$

be such a trail. If it is not a path, then $x_j = x_i$ for some $j < i$, i.e., the shortest trail is given as

$$\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{j-1}, x_j\}, \{x_j, x_{j+1}\}, \dots, \{x_i, x_{i+1}\}, \dots, \{x_{n-1}, x_n\}.$$

Here $x_j = x_i$, thus we can make an x - x' -trail shorter:

$$\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{j-1}, x_j\}, \{x_i, x_{i+1}\}, \dots, \{x_{n-1}, x_n\}.$$

This completes the proof. □

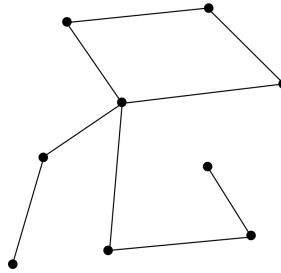


Fig. 3. Finding a shortest trail

A graph G is *path-connected* if any two vertices are connected by a trail (and, consequently, by a path). Also we say that vertices $x, x' \in V$ are in the same path component of G if there exists an x - x' -trail. This relation on the set of vertices is an equivalence relation. Indeed:

- There is always a *trivial* x - x -trail, i.e., $x \sim x$ (reflexivity).
- If there is an x - x' -trail, then there is x' - x -trail, i.e., if $x \sim x'$, then $x' \sim x$ (symmetry).
- If there is an x - x' -trail and there is an x' - x'' -trail, then there is x - x'' -trail, i.e., if $x \sim x'$ and $x' \sim x''$, then $x \sim x''$ (transitivity).

This equivalence relation breaks a graph G into path-components: $G = G_1 \sqcup \dots \sqcup G_s$, such that each graph G_i is path-connected.

We say that two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there are bijections $\phi : V \rightarrow V'$ and $\Phi : E \rightarrow E'$, such that for each edge $e = \{v, v'\} \in E$,

$$\Phi(e) = \Phi(\{v, v'\}) = \{\phi(v), \phi(v')\}.$$

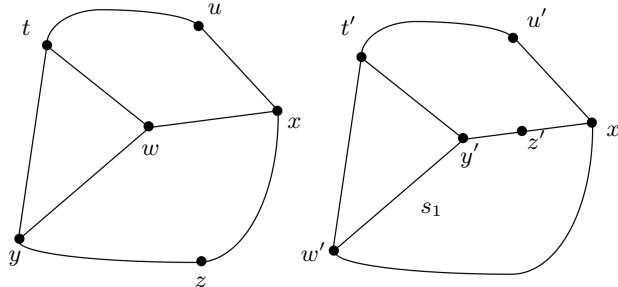


Fig. 4. Two isomorphic graphs

It is often useful to count the number of edges attached to a particular vertex. To get the right count, we need to treat loops differently from edges with two distinct vertices. We define $\deg(v)$, the degree of the vertex $v \in V(G)$,

to be the number of 2-vertex edges with v as a vertex plus twice the number of loops with v as vertex. If you think of a picture of G as being like a road map, then the degree of v is simply the number of roads you can take to leave v , with each loop counting as two roads.

The number $D_k(G)$ of vertices of degree k in G is an isomorphism invariant, as is the degree sequence $(D_0(G), D_1(G), D_2(G), \dots)$.

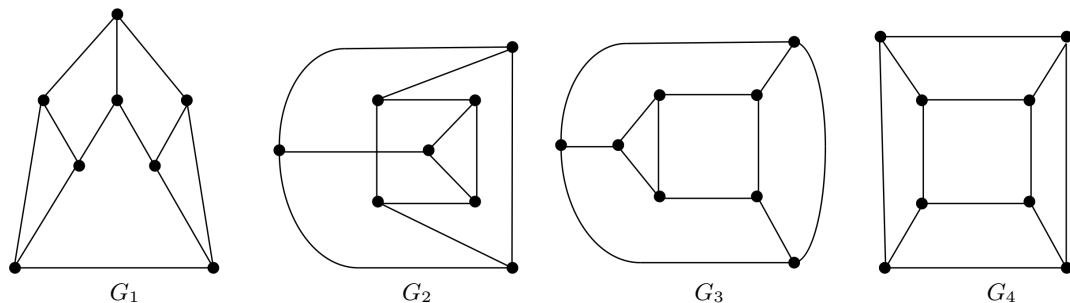


Fig. 5. Isomorphic and non-isomorphic graphs

Exercise 1. Find particular isomorphisms for the graphs G_1 , G_2 and G_3 from Fig. 5. Show that the graphs G_1 , G_2 and G_3 are not isomorphic to the graph G_4 .

Remark. Notice that the graphs G_1 , G_2 , G_3 and G_4 have the same number of vertices of the same degree. Thus having the same degree sequences does not guarantee that graphs are isomorphic.

Let n be a positive integer. Then a *complete graph* K_n is a graph with n vertices v_1, \dots, v_n and $\binom{n}{2}$ edges $e_{i,j} = \{v_i, v_j\}$, where $i \neq j$.

A complete graph K_n , contains subgraphs isomorphic to the graphs K_m for $m = 1, 2, \dots, n$. Such a subgraph can be obtained by selecting any m of the n vertices and using all the edges in K_n joining them. Thus K_5 contains $5 \text{ choose } 2 = 10$ subgraphs isomorphic to K_2 , $\binom{5}{3} = 10$ subgraphs isomorphic to K_3 [i.e., triangles], and $\binom{5}{4} = 5$ subgraphs isomorphic to K_4 . In fact, every graph with n or fewer vertices and with no loops or parallel edges is isomorphic to a subgraph of K_n ; just delete the unneeded edges from K_n .

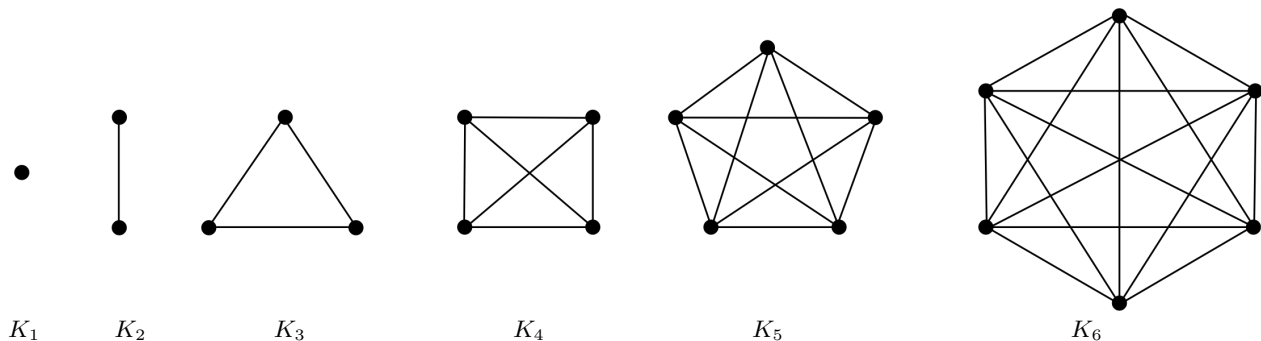


Fig. 6. Complete graphs

Complete graphs have a high degree of symmetry. Each permutation α of the vertices of a complete graph gives an isomorphism of the graph onto itself, since both $\{u, v\}$ and $\{\alpha(u), \alpha(v)\}$ are edges whenever $u \neq v$. The next theorem relates the degrees of vertices to the number of edges of the graph.

Theorem 2. *The sum of the degrees of the vertices of a graph $G = (V, E)$ is twice the number of edges, i.e.,*

$$\sum_{v \in V} \deg(v) = 2 \cdot |E(G)|.$$

Proof. Each edge, whether a loop or not, contributes 2 to the degree sum. This is a place where our convention that each loop contributes 2 to the degree of a vertex pays off. \square