

Summary on Lecture 3, July 22, 2015

The Method of Generating Functions: Examples.

Before analyzing the examples, we need one more identity on generation function:

$$\frac{1}{(1+x)^k} = \sum_{n=0}^{\infty} (-1)^n \binom{k+n-1}{n} x^n \quad (1)$$

This could be verified using Taylor-Maclaurin decomposition of the function $\frac{1}{(1+x)^k}$.

Example 1. Let $a_0 = 1$, and $a_n - 3a_{n-1} = n$, $n \geq 1$. This is new type of recurrence relations: *non-homogeneous*. We write the first few terms:

$$\begin{aligned} a_1 - 3a_0 &= 1 \\ a_2 - 3a_1 &= 2 \\ a_3 - 3a_2 &= 3 \\ \dots &\dots \\ a_n - 3a_{n-1} &= n \\ \dots &\dots \end{aligned}$$

We multiply the first equation by x , the second, by x^2 , the third, by x^3 and so on. We get:

$$\begin{aligned} a_1x - 3a_0x &= x \\ a_2x^2 - 3a_1x^2 &= x^2 \\ a_3x^3 - 3a_2x^3 &= 3x^3 \\ \dots &\dots \\ a_nx^n - 3a_{n-1}x^n &= nx^n \\ \dots &\dots \end{aligned}$$

We add the terms of the equations to get the identity:

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} n x^n. \quad (2)$$

We denote $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} a_n x^n - a_0 = f(x) - 1, \\ \sum_{n=1}^{\infty} a_{n-1} x^{n-1} &= f(x). \end{aligned}$$

Also, we recall from Example 4 (Lecture 4), we have

$$\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + \dots + n x^n + \dots = \frac{x}{(1-x)^2}$$

Then the identity (5) becomes

$$(f(x) - 1) - 3x f(x) = \frac{x}{(1-x)^2} \quad (3)$$

Then the identity (3) becomes

$$\begin{aligned}
 (f(x) - 1) - 3xf(x) &= -1 + f(x)(1 - 3x) = \frac{x}{(1-x)^2}, \quad \text{or} \\
 f(x)(1 - 3x) &= \frac{x}{(1-x)^2} + 1, \quad \text{or} \\
 f(x) &= \frac{x}{(1-x)^2(1-3x)} + \frac{1}{(1-3x)}
 \end{aligned} \tag{4}$$

Now we would like to decompose the term $\frac{x}{(1-x)^2(1-3x)}$ as the sum using so called method of partial fractions:

$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-3x)}$$

We obtain:

$$\begin{aligned}
 x &= A(1-x)(1-3x) + B(1-3x) + C(1-x)^2 \\
 &= 3Ax^2 - 4Ax + A + B - 3Bx + C - 2Cx + Cx^2 \\
 &= (3A + C)x^2 + (-4A - 3B - 2C)x + (A + B + C)
 \end{aligned}$$

We get the system:

$$\begin{cases} 3A + C = 0 \\ -4A - 3B - 2C = 1 \\ A + B + C = 0 \end{cases} \quad \text{or} \quad \begin{cases} C = -3A \\ -4A - 3B + 6A = 1 \\ A + B - 3A = 0 \end{cases} \quad \text{or} \quad \begin{cases} C = -3A \\ 2A - 3B = 1 \\ -2A + B = 0 \end{cases}$$

The equations

$$\begin{cases} 2A - 3B = 1 \\ -2A + B = 0 \end{cases}$$

give $-2B = 1$, i.e. $B = -\frac{1}{2}$, and $A = \frac{1}{2}B = -\frac{1}{4}$. Then the equation $C = -3A$ gives $C = \frac{3}{4}$. We obtain:

$$\begin{aligned}
 f(x) &= -\frac{1}{4} \cdot \frac{1}{(1-x)} - \frac{1}{2} \cdot \frac{1}{(1-x)^2} + \frac{3}{4} \cdot \frac{1}{(1-3x)} + \frac{1}{(1-3x)} \\
 &= -\frac{1}{4} \cdot \frac{1}{(1-x)} - \frac{1}{2} \cdot \frac{1}{(1-x)^2} + \frac{7}{4} \cdot \frac{1}{(1-3x)}
 \end{aligned}$$

Then we have

$$\begin{aligned}
 -\frac{1}{4} \cdot \frac{1}{(1-x)} &= -\frac{1}{4}(1 + x + x^2 + x^3 + \dots + x^n + \dots) = \sum_{n=0}^{\infty} \left(-\frac{1}{4}x^n\right) \\
 -\frac{1}{2} \cdot \frac{1}{(1-x)^2} &= -\frac{1}{2} \sum_{n=0}^{\infty} \left((-1)^n \binom{2+n-1}{n} (-x)^n\right) = \sum_{n=0}^{\infty} \left(-\frac{1}{2} \binom{2+n-1}{n} x^n\right) \\
 \frac{7}{4} \cdot \frac{1}{(1-3x)} &= \frac{7}{4} \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} \left(\frac{7 \cdot 3^n}{4} x^n\right)
 \end{aligned}$$

We notice that $\binom{2+n-1}{n} = \binom{n+1}{n} = \binom{n+1}{1} = (n+1)$. Then we add all three series together to get

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{4} - \frac{n+1}{2} + \frac{7 \cdot 3^n}{4}\right) x^n = \sum_{n=0}^{\infty} \left(-\frac{3}{4} - \frac{n}{2} + \frac{7 \cdot 3^n}{4}\right) x^n.$$

We obtain the answer: $a_n = -\frac{3}{4} - \frac{n}{2} + \frac{7 \cdot 3^n}{4} = \frac{7 \cdot 3^n - 3}{4} - \frac{n}{2}$.

Remark. We notice that for $n \geq 1$, the coefficient $\frac{3 \cdot 7(3^{n-1} - 1)}{4} - \frac{n}{2}$ is always an integer. Prove it!

Example 2. Let $a_0 = 3$, $a_1 = 7$ and $a_{n+2} - 5a_{n+1} + 6a_n = 2$, $n \geq 2$. This is also a *non-homogeneous* new type of recurrence relation. We multiply the relation by x^{n+2} :

$$a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_nx^{n+2} = 2x^{n+2}$$

and we obtain the identity

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5 \sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6 \sum_{n=0}^{\infty} a_nx^{n+2} = 2 \sum_{n=0}^{\infty} x^{n+2}. \quad (5)$$

We denote $f(x) = \sum_{n=0}^{\infty} a_nx^n$, then we have:

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = f(x) - a_0 - a_1x = f(x) - 3 - 7x$$

$$\sum_{n=0}^{\infty} a_{n+1}x^{n+2} = x(f(x) - a_0) = x(f(x) - 3)$$

$$\sum_{n=0}^{\infty} a_nx^{n+2} = x^2f(x)$$

$$\sum_{n=0}^{\infty} x^{n+2} = \frac{x^2}{1-x}$$

Then the identity (5) turns to the equation for $f(x)$:

$$(f(x) - 3 - 7x) - 5x(f(x) - 3) + 6x^2f(x) = \frac{2x^2}{1-x} \quad \text{or}$$

$$f(x)(1 - 5x + 6x^2) = \frac{2x^2}{1-x} + 3 - 8x$$

Thus we obtain:

$$\begin{aligned} f(x) &= \frac{(3 - 8x)(1 - x) + 2x^2}{(1 - x)(1 - 5x + 6x^2)} \\ &= \frac{3 - 11x + 10x^2}{(1 - x)(1 - 5x + 6x^2)} \\ &= \frac{(3 - 5x)(1 - 2x)}{(1 - x)(1 - 2x)(1 - 3x)} = \frac{3 - 5x}{(1 - x)(1 - 3x)} \end{aligned}$$

Here we used that $3 - 11x + 10x^2 = (3 - 5x)(1 - 2x)$ and $1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$. Now we write

$$\frac{3 - 5x}{(1 - x)(1 - 3x)} = \frac{A}{1 - x} + \frac{B}{1 - 3x} = \frac{A - 3Ax + B - Bx}{(1 - x)(1 - 3x)} = \frac{(A + B) - (3A + B)x}{(1 - x)(1 - 3x)}$$

which yields the system:

$$\begin{cases} A + B = 3 \\ 3A + B = 5 \end{cases} \quad \text{or} \quad \begin{cases} A = 1 \\ B = 2 \end{cases}$$

This gives the generating function:

$$f(x) = \frac{1}{1-x} + \frac{2}{1-3x} = \sum_{n=0}^{\infty} x^n + 2 \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (1 + 2 \cdot 3^n) x^n.$$

We obtain the solution: $a_n = 1 + 2 \cdot 3^n$, $n \geq 0$.

Example 3. Let $a_0 = 0$, $a_1 = 6$ and $a_n = -3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n$, $n \geq 2$. This is also a *non-homogeneous* new type of recurrence relation. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function. We multiply by x^n the recurrence relation to get

$$\sum_{n=2}^{\infty} a_n x^n = -3 \sum_{n=2}^{\infty} a_{n-1} x^n + 10 \cdot \sum_{n=2}^{\infty} a_{n-2} x^n + 3 \cdot \sum_{n=2}^{\infty} 2^n x^n$$

We notice:

$$\sum_{n=2}^{\infty} a_n x^n = f(x) - a_1 x - a_0 = f(x) - 6x$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = x(f(x) - a_0) = xf(x)$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 f(x)$$

$$\sum_{n=2}^{\infty} 2^n x^n = \frac{1}{1-2x} - (1+2x)$$

Then we obtain

$$f(x) - 6x = -3xf(x) + 10x^2 + \frac{3}{1-2x} - 3(1+2x)$$

We simplify:

$$f(x)(1+3x-10x^2) = \frac{3}{1-2x} - 3 = \frac{6x}{1-2x}$$

Since $1+3x-10x^2 = (1+5x)(1-2x)$, we obtain

$$f(x) = \frac{6x}{(1-2x)^2(1+5x)}.$$

We use the same method as above:

$$\frac{6x}{(1-2x)^2(1+5x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+5x}$$

We obtain:

$$\begin{aligned} 6x &= A(1-2x)(1+5x) + B(1+5x) + C(1-2x)^2 \\ &= (A+B+C) + (3A+5B-4C)x + (-10A+4C)x^2 \end{aligned}$$

We solve the system

$$\begin{cases} A+B+C &= 0 \\ 3A+5B-4C &= 6 \\ -10A+4C &= 0 \end{cases} \implies \begin{cases} A &= -\frac{12}{49} \\ B &= \frac{6}{7} \\ C &= -\frac{30}{49} \end{cases}$$

We obtain:

$$\begin{aligned}f(x) &= -\frac{12}{49} \cdot \frac{1}{1-2x} + \frac{6}{7} \cdot \frac{1}{(1-2x)^2} - \frac{30}{49} \cdot \frac{1}{1+5x} \\&= -\frac{12}{49} \cdot \sum_{n=0}^{\infty} 2^n x^n + \frac{6}{7} \cdot \sum_{n=0}^{\infty} (n+1)2^n x^n - \frac{30}{49} \cdot \sum_{n=0}^{\infty} 5^n x^n \\&= \sum_{n=0}^{\infty} \left(-\frac{12}{49} \cdot 2^n + \frac{6}{7} \cdot (n+1)2^n - \frac{30}{49} \cdot 5^n \right) x^n\end{aligned}$$

The answer: $a_n = -\frac{12}{49} \cdot 2^n + \frac{6}{7} \cdot (n+1)2^n - \frac{30}{49} \cdot 5^n$.

Exercise 1. Let $a_0 = 0$, $a_1 = 6$, and $a_n = -3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n$ for $n \geq 2$. Use generating functions to find a formula for a_n .

Exercise 2. Let $a_0 = 1$, and $a_n = a_{n-1} + n$ for $n \geq 2$. Use generating functions to find a formula for a_n .¹

Exercise 3. Let $a_0 = 0$, and $a_n = 3a_{n-1} + 7$ for $n \geq 2$. Use generating functions to find a formula for a_n .

Exercise 4. For each of the following functions $G(n)$ use generating functions to find a formula for a_n , where $a_{n+2} - 6a_{n+1} - 7a_n = G(n)$.

(a) $G(n) = (-5)^n \cdot 24$, $a_0 = 3$, $a_1 = -1$.

(b) $G(n) = 12n^2 - 4n + 10$, $a_0 = 0$, $a_1 = -10$.

¹The answer is $a_n = 1 + \binom{n+1}{2}$.