

Summary on Lecture 2, July 21, 2015

We continue with Fibonacci numbers and application for the Euclidian algorithm.

Let m, k be positive integers, $m \geq k \geq 2$, and we look at the division:

$$m = q \cdot k + r, \quad 0 \leq r < k.$$

Recall that a key to compute $\gcd(m, k)$ is the identity $\gcd(m, k) = \gcd(k, r)$. We organize the Euclidian Algorithm as follows to match the notations from the book.

Let $r_0 = m, r_1 = k$. Then we have the divisions:

$$\begin{aligned} r_0 &= q_1 r_1 + r_2 & 0 \leq r_2 < r_1 \\ r_1 &= q_2 r_2 + r_3 & 0 \leq r_3 < r_2 \\ r_2 &= q_3 r_3 + r_4 & 0 \leq r_4 < r_3 \\ \dots & \dots & \dots \\ r_{n-2} &= q_{n-1} r_{n-1} + r_n & 0 \leq r_n < r_{n-1} \\ r_{n-1} &= q_n r_n & \end{aligned} \tag{1}$$

Then we have the sequence of identities:

$$\gcd(m, k) = \gcd(r_0, r_1) = \gcd(r_1, r_2) = \gcd(r_2, r_3) = \dots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n.$$

We notice that we have performed n divisions, and every quotient $q_i \geq 1$ for all $i = 1, 2, \dots, n - 1$. Then the $r_{n-1} = q_n r_n$ and $r_n < r_{n-1}$ imply that $q_n \geq 2$.

Now we examine the remainders $r_n, r_{n-1}, \dots, r_2, r_1$ (here $r_1 = k$). We have:

$$\begin{aligned} r_n > 0, \text{ i.e. } r_n \geq 1 \text{ thus } r_n &\geq F_2 = 1 & \text{i.e. } r_n &\geq F_2 \\ q_n \geq 2 \text{ and } r_n \geq 1 \text{ thus } r_{n-1} = q_n r_n &\geq 2 \cdot 1 = 2 = F_3 & \text{i.e. } r_{n-1} &\geq F_3 \\ r_{n-2} = q_{n-1} r_{n-1} + r_n \geq 1 \cdot r_{n-1} + r_n &\geq F_2 + F_3 = F_4 & \text{i.e. } r_{n-2} &\geq F_4 \\ r_{n-3} = q_{n-2} r_{n-2} + r_{n-1} \geq 1 \cdot r_{n-2} + r_{n-1} &\geq F_3 + F_4 = F_5 & \text{i.e. } r_{n-3} &\geq F_5 \\ \dots & \dots & \dots & \dots \\ r_2 = q_3 r_3 + r_4 \geq 1 \cdot r_3 + r_4 &\geq F_{n-1} + F_{n-2} = F_n & \text{i.e. } r_2 &\geq F_n \\ r_1 = q_2 r_2 + r_3 \geq 1 \cdot r_2 + r_3 &\geq F_n + F_{n-1} = F_{n+1} & \text{i.e. } r_1 &\geq F_{n+1} \end{aligned}$$

Since $k = r_1$, we obtain $k \geq F_{n+1}, m \geq k \geq 2$. Lemma 1 then implies that

$$k \geq F_{n+1} \geq \alpha^{n+1-2} = \alpha^{n-1}, \text{ or } \log_{10} k \geq (n - 1) \log_{10} \alpha$$

Then we have that $\log_{10} \alpha = \log_{10} \left(\frac{1+\sqrt{5}}{2}\right) = 0.208\dots > 0.2 = \frac{1}{5}$, i.e., $\log_{10} k \geq \frac{n-1}{5}$. This means that if k is such that $10^{s-1} \leq k < 10^s$, then

$$s = \log_{10} 10^s > \log_{10} k \geq \frac{n-1}{5}, \text{ or } n < 5s + 1.$$

We proved the following result.

Theorem 3. Let $m \geq k \geq 2$, and k has at most s digits, (i.e., $10^{s-1} \leq k < 10^s$). Then the Euclidian Algorithm requires at most $5s$ divisions to compute $\gcd(m, k)$.

Second Order Recurrence Relations (continuation)

Example: legal arithmetic expressions without parenthesis. In most computing languages, it important to use “legal arithmetic expressions without parenthesis”. These expressions are made up out of the digits $0, 1, \dots, 9$ and binary symbols $+, *, /$. For example, the expressions $7+8$, $5+7*3$, $33*7+4+6*4$ are legal expressions, and the expressions $/7+8$, $5+7*3+$, $33*7+/4+6*4$ are not.

We denote by a_n the number of legal expressions of length n . Then $a_1 = 10$ since the only legal expressions of length 1 are the digits $0, 1, \dots, 9$. Then $a_2 = 100$ which accounts for the expressions $00, 01, \dots, 99$.

Let $n \geq 3$. We observe:

(1) Let x be an arithmetic legal expression of $(n-1)$ symbols. Then the last symbol must be a digit. We add one more digit to the right of x and obtain $10x$ more legal expressions of the length n .

(2) Let y be an arithmetic legal expression of $(n-2)$ symbols. Then we can add to the right of y one of the following 29 2-symbol expressions: $+0, +1, \dots, +9, *0, *1, \dots, *9, /1, \dots, /9$ (no division by 0 is allowed).

We obtain the recurrence relation: $a_0 = 10, a_1 = 10, a_n = 10a_{n-1} + 29a_{n-2}$ for $n \geq 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 10, a_1 = 10, a_n = 10a_{n-1} + 29a_{n-2}, n \geq 3$.

Example. We would like to find a number of binary sequences of the length n without any consecutive 0's.

Let a_n denote the number of such sequences of length $n \geq 1$. Clearly, if $n = 1$, we have 0, 1, i.e., $a_1 = 2$, if $n = 2$, we have the sequences 01, 10, 11, i.e., $a_2 = 3$.

Let $n \geq 3$. Let $x_1 \cdots x_{n-2}x_{n-1}x_n$ be a sequence like that. There are two cases:

(1) The last symbol $x_n = 1$. Then the sequence $x_1 \cdots x_{n-2}x_{n-1}$ has no consecutive 0's.

(2) The last symbol $x_n = 0$. Then $x_{n-1} = 1$, and the sequence $x_1 \cdots x_{n-2}$ has no consecutive 0's.

Thus we conclude that $a_n = a_{n-1} + a_{n-2}$. Also we notice that the initial conditions $a_1 = 2, a_2 = 3$ could be replaced by $a_0 = 1, a_1 = 2$. Then $a_2 = a_1 + a_0 = 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 1, a_1 = 2, a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

The case of complex roots. Let $z = x + iy \in \mathbf{C}$ be a complex number. Then we let $|z| = \sqrt{x^2 + y^2}$, and we can write z as

$$z = |z|(\cos \theta + i \sin \theta), \quad \cos \theta = \frac{x}{|z|}, \quad \sin \theta = \frac{y}{|z|}.$$

There is a standard notation $e^{i\theta} := \cos \theta + i \sin \theta$. There is a remarkable formula (DeMoivre Theorem):

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \quad \text{or} \quad (e^{i\theta})^n = e^{in\theta}$$

We prove it by induction. Clearly this formula holds for $n = 1$. Assume it holds for $n = k$. Then we have:

$$\begin{aligned} (\cos \theta + i \sin \theta)^{k+1} &= (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \\ &= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta) \\ &= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\cos k\theta \sin \theta + \sin k\theta \cos \theta) \\ &= \cos(k+1)\theta + i \sin(k+1)\theta. \end{aligned}$$

Here we used the formulas:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Example. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} - 2a_{n-2}$. Then again, we are looking for a solution as $a_n = cr^n$, $c \neq 0$. We have substitute $a_n = cr^n$ to our recurrence relation:

$$cr^n = 2cr^{n-1} - 2cr^{n-2} \quad \text{or} \quad r^2 - 2r + 2 = 0.$$

We find the solutions of the characteristic equation:

$$r_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm i.$$

Then we have:

$$r_1 = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

$$r_2 = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right).$$

Now we are looking for a solution in the form $a_n = c_1r_1^n + c_2r_2^n$. We notice the following:

$$\begin{aligned} a_n &= c_1(1+i)^n + c_2(1-i)^n \\ &= c_1\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n + c_2\left(\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right)^n \\ &= c_1\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + c_2\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \\ &= \left(\sqrt{2}\right)^n\left(K_1\cos\frac{n\pi}{4} + K_2\sin\frac{n\pi}{4}\right), \end{aligned}$$

where $K_1 = c_1 + c_2$, $K_2 = i(c_1 - c_2)$. Clearly we would like to find **real values** of K_1 and K_2 . We substitute $n = 0$ and $n = 1$ to get the system:

$$\begin{cases} K_1 \cos 0 + K_2 \sin 0 & = a_0 = 1 \\ \sqrt{2}(K_1 \cos \frac{\pi}{4} + K_2 \sin \frac{\pi}{4}) & = a_1 = 2 \end{cases} \quad \text{or} \quad \begin{cases} K_1 & = 1 \\ K_1 + K_2 & = 2 \end{cases} \quad \text{or} \quad \begin{cases} K_1 & = 1 \\ K_2 & = 1 \end{cases}$$

We obtain the answer:

$$a_n = \left(\sqrt{2}\right)^n \left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)$$

General Case. Now we assume that we have a second order recurrence relation, i.e. a_0, a_1 , are given and $a_n = Aa_{n-1} + Ba_{n-2}$, $n \geq 2$, where the coefficients A and B are real numbers (in fact, they are integers in all our examples). Then the characteristic equation is given as $r^2 - Ar - B = 0$. Assume that the roots r_1, r_2 are complex. Obviously, it means that r_1 and r_2 are conjugate, i.e., we can write $r_1 = \rho(\cos\theta + i\sin\theta)$, and $r_2 = \rho(\cos\theta - i\sin\theta)$. Thus we may look for a solution for a_n in the form:

$$\begin{aligned} a_n &= c_1r_1^n + c_2r_2^n \\ &= c_1\rho^n(\cos n\theta + i\sin n\theta) + c_2\rho^n(\cos n\theta - i\sin n\theta) \\ &= \rho^n((c_1 + c_2)\cos n\theta + i(c_1 - c_2)\sin n\theta) \\ &= \rho^n(K_1\cos n\theta + K_2\sin n\theta). \end{aligned}$$

Here $K_1 = (c_1 + c_2)$ and $K_2 = i(c_1 - c_2)$. In particular, it means that the expression $a_n = \rho^n(K_1\cos n\theta + K_2\sin n\theta)$ satisfies the recurrence relation we started with. We also notice that K_1 and K_2 are assumed to be real. Since a_0 and a_1 are given, we find them by substituting $n = 0$ and $n = 1$:

$$\begin{cases} K_1 \cos 0 + K_2 \sin 0 & = a_0 \\ \rho(K_1 \cos \theta + K_2 \sin \theta) & = a_1 \end{cases} \quad \text{or} \quad \begin{cases} K_1 & = a_0 \\ \rho(K_1 \cos \theta + K_2 \sin \theta) & = a_1 \end{cases}$$

We notice that the system always has a solution provided $\rho \sin \theta \neq 0$ for arbitrary initial coefficients a_0 and a_1 . On the other hand, the condition $\rho \sin \theta = 0$ means that either $\rho = 0$ or $\sin \theta = 0$. Each of those imply that the roots r_1, r_2 are real.

We summarize the above discussion:

Theorem 3. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero real constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two complex roots

$$r_1 = \rho(\cos \theta + i \sin \theta), \quad r_2 = \rho(\cos \theta - i \sin \theta),$$

Then $a_n = \rho^n(K_1 \cos n\theta + K_2 \sin n\theta)$, where the coefficients K_1, K_2 are determined by solving the system

$$\begin{cases} K_1 & = a_0 \\ \rho(K_1 \cos \theta + K_2 \sin \theta) & = a_1 \end{cases}$$

The Method of Generating Functions.

There is another powerful technique to resolve recurrence relations. Let $a_0, a_1, \dots, a_n, \dots$ be a sequence of real numbers. Then the series

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

is called a *generating function* for the sequence $\{a_i\}$.

Examples. (1) The function

$$f(x) = (1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

is a generating function for the sequence $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}, 0, \dots$

(2) We notice that $1 - x^{n+1} = (1-x)(1+x+x^2+\dots+x^n)$. This gives the generating function

$$\frac{1-x^{n+1}}{1-x}$$

for the sequence $1, 1, \dots, 1, 0, 0, \dots$

(3) Similarly to the previous example, we notice that $1 = (1-x)(1+x+x^2+\dots+x^n+\dots)$. This gives the generating function

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

for the sequence $1, 1, \dots, 1, \dots$

(4) Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx} \frac{1}{1-x} = \sum_{i=0}^{\infty} \frac{d}{dx} x^i$$

Since $\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$, we obtain the identity:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1} + \dots$$

Thus the function $\frac{1}{(1-x)^2}$ is a generating function for the sequence $1, 2, 3, 4, \dots, n, \dots$. We also notice that the function

$$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n + \dots$$

is a generating function for the sequence $0, 1, 2, 3, 4, \dots, n, \dots$

(5) We take one more derivative: Now we take a derivative of both sides of the generating function:

$$\frac{d}{dx} \frac{x}{(1-x)^2} = \frac{d}{dx} (0 + x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n + \dots)$$

Since $\frac{d}{dx} \frac{x}{(1-x)^2} = \frac{x+1}{(1-x)^3}$, we obtain the identity:

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots + n^2x^{n-1} + \dots$$

Thus the function $\frac{x+1}{(1-x)^3}$ is a generating function for the sequence $1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots$. Then we see that the function

$$\frac{x(x+1)}{(1-x)^3} = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots + n^2x^n + \dots$$

is a generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots$