Summary on Lecture 1, July 20, 2015

Recurrence Relations

Warm-up: linear reccurence relations.

(1) **Geometric progression.** Define a sequence $\{a_n\}$ as follows: $a_0 = A$, $a_{n+1} = da_n$, $n \geq 1$. Then we have:

$$a_1 = dA, \quad a_2 = d^2A, \quad a_3 = d^3A, \ \ldots \ a_n = d^nA, \ \ldots$$

Thus we have a general formula: $a_n = d^nA$. This is a *geometric progression.*

Exercise. Prove formula $a_n = d^nA$ by induction.

Definition. A reccurence relation $a_{n+1} - da_n = 0$, where d is a constant, is called *linear relation.* More general, a reccurence relation $a_{n+1} - da_n = f(n)$, where c is a constant, and $f(n)$ is a function, is called a *first order relation.*

(2) **Example: Bubble Sort algorithm.** Let $x_1, \ldots, x_n$ be n real numbers. We would like to sort them out into ascending order. Here is an algorithm known as **BubbleSort**:

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begin(BubbleSort)
  for i := 1 to n-1 do
    for j := n down to i+1 do
      if x_j < x_{j-1} then
        begin(Interchange)
          t := x_{j-1}
          x_{j-1} := x_j
          x_j := t
        end(Interchange)
end(BubbleSort)
```

First, we would like to understand how does it work. Let us start with the sequence $(x_1, x_2, x_3, x_4, x_5) = (7, 9, 2, 5, 8)$.

$i=1$		$j=5$	$j=4$	$j=3$	$j=2$	
x_1		7	7	7	2	2
x_2		9	9	2	7	7
x_3	:=	2	2	9	9	9
x_4		5	5	5	5	5
x_5		8	8	8	8	8

$i=2$		$j=5$	$j=4$	$j=3$	
x_1		2	2	2	2
x_2		7	7	5	5
x_3	:=	9	5	7	7
x_4		5	9	9	9
x_5		8	8	8	8

$i=3$		$j=5$	$j=4$	
x_1		2	2	2
x_2		5	5	5
x_3	:=	7	7	7
x_4		8	8	8
x_5		9	9	9

$i=4$		$j=5$	
x_1		2	2
x_2		5	5
x_3	:=	7	7
x_4		8	8
x_5		9	9

Here we have: for $i = 1$, 4 comparisons and 2 interchanges, for $i = 2$, 3 comparisons and 2 interchanges, for $i = 3$, 2 comparisons and 1 interchange, for $i = 4$, 1 comparison and no interchanges.

Now we denote by a_n a total number of comparisons to sort out a sequence $(x_1, \ldots, x_n)$. First, we can identify the smallest number: this is done when we run the algorithm for $i = 1$. Clearly, we use $(n-1)$ comparisons for that. Then we obtain the recursion:

$$a_1 = 0, \quad a_n = a_{n-1} + (n-1).$$

We have:

$$\begin{array}{lcllcl} a_1 & = & 0 & & & \\ a_2 & = & a_1 + (2-1) & & = & 1 \\ a_3 & = & a_2 + (3-1) & & = & 1+2 \\ a_4 & = & a_3 + (4-1) & & = & 1+2+3 \\ \cdots & & \cdots & & & \cdots \\ a_n & = & a_{n-1} + (n-1) & & = & 1+2+3+\cdots+(n-1) \end{array}$$

The answer:

$$a_n = 1+2+3+\cdots+(n-1) = \frac{(n-1)n}{2} = \frac{1}{2}(n^2-n).$$

In that case we say that the time-complexity function of that algorithm is $O(n^2)$.

Second Order Recurrence Relations. Let $\{a_n\}$ be a Fibonacci sequence, i.e. $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. We would like to find a *closed formula* for a_n's. Let us try $a_n = c \cdot r^n$, where $c \neq 0$ and r some real numbers. Then the relation $a_n = a_{n-1} + a_{n-2}$ gives:

$$cr^n = cr^{n-1} + cr^{n-2}, \quad n \geq 2.$$

We cancel cr^{n-2} and get the equation $r^2 = r+1$ or $r^2 - r - 1 = 0$. We find the solutions:

$$r = \frac{1 \pm \sqrt{5}}{2}, \quad \text{or} \quad r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

Then both sequences $c_1 r_1^n$ and $c_2 r_2^n$ will satisfy the relation $a_n = a_{n-1} + a_{n-2}$. Moreover, the sequence $c_1 r_1^n + c_2 r_2^n$ will satisfy the same relation. The we can find c_1 and c_2.

We have for $n = 0$ and $n = 1$:

$$\left\{\begin{array}{lcl} 0 & = & c_1 + c_2 \\ 1 & = & c_1 r_1 + c_2 r_2 \end{array}\right. \qquad \left\{\begin{array}{lcl} c_2 & = & -c_1 \\ 1 & = & c_1 r_1 - c_1 r_2 \end{array}\right. \qquad \left\{\begin{array}{lcl} c_2 & = & -\frac{1}{r_1 - r_2} \\ c_1 & = & \frac{1}{r_1 - r_2} \end{array}\right.$$

Since $r_1 - r_2 = \sqrt{5}$, we obtain a formula for a_n:

$$a_n = c_1 r_1^n + c_2 r_2^n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Let $a_n = Aa_{n-1} + Ba_{n-2}$ be a second order recurrence relation. Then the equation $r^2 - Ar - B = 0$ is called a *characteristic equation* of that relation.

Theorem 1. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a second order recurrence relation, $n \geq 2$, where A, B are non-zero constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two real different real solutions r_1 and r_2. Then $a_n = c_1 r_1^n + c_2 r_2^n$, where the constants c_1 and c_2 are determined by solving the system

$$\left\{\begin{array}{lcl} a_0 & = & c_1 + c_2 \\ a_1 & = & c_1 r_1 + c_2 r_2 \end{array}\right.$$

Proof. Indeed, we look for a solution $a_n = cr^n$, then the recurrence realtion $a_n = Aa_{n-1} + Ba_{n-2}$ gives the characteristic equation $r^2 - Ar - B = 0$. By assumption, there are two two different real solutions, r_1 and r_2 of $r^2 - Ar - B = 0$. Then the sum $c_1 r_1^n + c_2 r_2^n$ will satisfy the recurrence. Finally, we notice that the system $\left\{\begin{array}{lcl} a_0 & = & c_1 + c_2 \\ a_1 & = & c_1 r_1 + c_2 r_2 \end{array}\right.$ always have a unique solution if $r_1 \neq r_2$ (Explain why).

Next question: How to solve this problem if $r_1 = r_2$?

Example. Consider the sequence defined by $a_0 = 1$, $a_1 = -3$, and $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. The we try $a_n = cr^n$ with $c \neq 0$ to get the following characteristic equation: $r^2 - 6r + 9 = 0$. We obtain the solution

$r = r_1 = r_2 = 3$. We notice that the $a_n = c_1 r^n + c_2 n r^n$ satisfies the relation $a_n = 6a_{n-1} - 9a_{n-2}$. We notice that $6 = 2r$ and $9 = r^2$. Then, indeed, we have:

$$\begin{aligned} c_1 r^n + c_2 n r^n &= 6c_1 r^{n-1} + 6c_2(n-1)r^{n-1} - 9c_1 r^{n-2} - 9c_2(n-2)r^{n-2} \\ &= c_1(6r^{n-1} - 9r^{n-2}) + c_2(2(n-1)r \cdot r^{n-1} - (n-2)r^2 r^{n-2}) \\ &= c_1(6r^{n-1} - 9r^{n-2}) + c_2 n r^n. \end{aligned}$$

This is true since $r^n = 6r^{n-1} - 9r^{n-2}$. Thus $a_n = c_1 r^n + c_2 n r^n = c_1 3^n + c_2 3^n$ satisfies the relation $a_n = 6a_{n-1} - 9a_{n-2}$. Then for $n = 0, 1$, we obtain:

$$\left\{ \begin{array}{rcl} 1 &=& c_1 \\ -3 &=& 3c_1 + 3c_2 \end{array} \right. \implies \left\{ \begin{array}{rcl} 1 &=& c_1 \\ -1 &=& 1 + c_2 \end{array} \right. \implies \left\{ \begin{array}{rcl} 1 &=& c_1 \\ -2 &=& c_2 \end{array} \right.$$

We obtain the answer $a_n = 3^n - 2n3^n$. This example is a particular case of the following Theorem:

Theorem 2. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has one real solution $r \neq 0$ (i.e., $r_1 = r_2 = r$) Then $a_n = c_1 r^n + c_2 n r^n$, where the constants c_1 and c_2 are determined by solving the system $\left\{ \begin{array}{rcl} a_0 &=& c_1 \\ a_1 &=& c_1 r + c_2 r \end{array} \right.$

Exercise: Prove Theorem 2.

Fibonacci numbers again: nontrivial application. Now we denote by F_n the Fibonnaci numbers defined above, i.e. $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Let $\alpha = \frac{1+\sqrt{5}}{2}$. We need the following property:

Lemma 1. $F_n > \alpha^{n-2}$ for $n \geq 3$.

Exercise: Prove Lemma 1 by induction.