Summary on Lecture 5, October 2, 2017

The case of complex roots. Let $z = x + iy \in \mathbf{C}$ be a complex number. Then we let $|z| = \sqrt{x^2 + y^2}$, and we can write z as

$$z = |z|(\cos\theta + i\sin\theta), \quad \cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}.$$

There is a standard notation $e^{i\theta} := \cos\theta + i\sin\theta$. There is a remarkable formula (DeMoivre Theorem):

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{or} \quad (e^{i\theta})^n = e^{in\theta}$$

We prove it by induction. Clearly this formula holds for $n = 1$. Assume it holds for $n = k$. Then we have:

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\[2mm]
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\[2mm]
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta) \\[2mm]
&= \cos(k+1)\theta + i\sin(k+1)\theta.
\end{aligned}
$$

Here we used the formulas:

$$
\begin{aligned}
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\[2mm]
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}
$$

Example. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} - 2a_{n-2}$. Then again, we are looking for a solution as $a_n = cr^n$, $c \neq 0$. We have substitute $a_n = cr^n$ to our recurrence relation:

$$cr^n = 2cr^{n-1} - 2cr^{n-2} \quad \text{or} \quad r^2 - 2r + 2 = 0.$$

We find the solutions of the characteristic equation:

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm i.$$

Then we have:

$$
\begin{aligned}
r_1 &= 1 + i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}\right) = \sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}), \\[2mm]
r_2 &= 1 - i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}\right) = \sqrt{2}(\cos\tfrac{\pi}{4} - i\sin\tfrac{\pi}{4}).
\end{aligned}
$$

Now we are looking for a solution in the form $a_n = c_1 r_1^n + c_2 r_2^n$. We notice the following:

$$
\begin{aligned}
a_n &= c_1(1+i)^n + c_2(1-i)^n \\[2mm]
&= c_1\left(\sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4})\right)^n + c_2\left(\sqrt{2}(\cos\tfrac{\pi}{4} - i\sin\tfrac{\pi}{4})\right)^n \\[2mm]
&= c_1\left(\sqrt{2}\right)^n\left(\cos\tfrac{n\pi}{4} + i\sin\tfrac{n\pi}{4}\right) + c_2\left(\sqrt{2}\right)^n\left(\cos\tfrac{n\pi}{4} - i\sin\tfrac{n\pi}{4}\right) \\[2mm]
&= \left(\sqrt{2}\right)^n\left(K_1\cos\tfrac{n\pi}{4} + K_2\sin\tfrac{n\pi}{4}\right),
\end{aligned}
$$

where $K_1 = c_1 + c_2$, $K_2 = i(c_1 - c_2)$. Clearly we would like to find **real values** of K_1 and K_2. We substitute $n = 0$ and $n = 1$ to get the system:

$$
\left\{
\begin{aligned}
K_1\cos 0 + K_2\sin 0 &= a_0 = 1 \\
\sqrt{2}(K_1\cos\tfrac{\pi}{4} + K_2\sin\tfrac{\pi}{4}) &= a_1 = 2
\end{aligned}
\right.
\quad \text{or} \quad
\left\{
\begin{aligned}
K_1 &= 1 \\
K_1 + K_2 &= 2
\end{aligned}
\right.
\quad \text{or} \quad
\left\{
\begin{aligned}
K_1 &= 1 \\
K_2 &= 1
\end{aligned}
\right.
$$

We obtain the answer:

$$a_n = \left(\sqrt{2}\right)^n \left(\cos \tfrac{n\pi}{4} + \sin \tfrac{n\pi}{4}\right)$$

General Case. Now we assume that we have a second order recurrence relation, i.e. a_0, a_1, are given and $a_n = Aa_{n-1} + Ba_{n-2}$, $n \geq 2$, where the coefficients A and B are real numbers (in fact, they are integers in all our examples). Then the characteristic equation is given as $r^2 - Ar - B = 0$. Assume that the roots r_1, r_2 are complex. Obviously, it means that r_1 and r_2 are conjugate, i.e., we can write $r_1 = \rho(\cos\theta + i\sin\theta)$, and $r_2 = \rho(\cos\theta - i\sin\theta)$. Thus we may look for a solution for a_n in the form:

$$
\begin{aligned}
a_n &= c_1 r_1^n + c_2 r_2^n \\[1mm]
&= c_1 \rho^n (\cos n\theta + i\sin n\theta) + c_2 \rho^n (\cos n\theta - i\sin n\theta) \\[1mm]
&= \rho^n ((c_1 + c_2)\cos n\theta + i(c_1 - c_2)\sin n\theta) \\[1mm]
&= \rho^n (K_1 \cos n\theta + K_2 \sin n\theta).
\end{aligned}
$$

Here $K_1 = (c_1+c_2)$ and $K_2 = i(c_1-c_2)$. In particular, it means that the expression $a_n = \rho^n(K_1 \cos n\theta + K_2 \sin n\theta)$ satisfies the recurrence relation we started with. We also notice that K_1 and K_2 are assumed to be real. Since a_0 and a_1 are given, we find them by substituting $n = 0$ and $n = 1$:

$$
\begin{cases}
K_1 \cos 0 + K_2 \sin 0 &= a_0 \\
\rho(K_1 \cos\theta + K_2 \sin\theta) &= a_1
\end{cases}
\quad \text{or} \quad
\begin{cases}
K_1 &= a_0 \\
\rho(K_1 \cos\theta + K_2 \sin\theta) &= a_1
\end{cases}
$$

We notice that the system always has a solution provided $\rho\sin\theta \neq 0$ for arbitrary initial coefficients a_0 and a_1. On the other hand, the condition $\rho\sin\theta = 0$ means that either $\rho = 0$ or $\sin\theta = 0$. Each of those imply that the roots r_1 r_2 are real.

We summarize the above discussion:

Theorem 3. Let a_0 and a_1 are given, and $a_n = Aa_{n-1} + Ba_{n-2}$ be a recurrence relation, $n \geq 2$, where A, B are non-zero real constants. Assume that the characteristic equation $r^2 - Ar - B = 0$ has two complex roots

$$r_1 = \rho(\cos\theta + i\sin\theta), \quad r_2 = \rho(\cos\theta - i\sin\theta),$$

Then $a_n = \rho^n(K_1 \cos n\theta + K_2 \sin n\theta)$, where the coefficients K_1, K_2 are determined by solving the system

$$
\begin{cases}
K_1 &= a_0 \\
\rho(K_1 \cos\theta + K_2 \sin\theta) &= a_1
\end{cases}
$$