Summary on Lecture 4, September 29, 2017

Second Order Recurrence Relations (continuation)

Example: legal arithmetic expressions without parenthesis. In most computing languages, it important to use "legal arithmetic expressions without parenthesis". These expressions are made up out of the digits 0,1,..., 9 and binary symbols $+, *, /$. For example, the expressions $7+8$, $5+7*3$, $33*7+4+6*4$ are legal expressions, and the expressions $/7+8$, $5+7*3+$, $33*7+/4+6*4$ are not.

We denote by a_n the number of legal expressions of length n. Then $a_1 = 10$ since the only legal expressions of length 1 are the digits $0, 1, \ldots, 9$. Then $a_2 = 100$ which accounts for the expressions 00, 01, $\ldots$, 99.

Let $n \geq 3$. We observe:

(1) Let x be an arithmetic legal expression of $(n-1)$ symbols. Then the last symbol must be a digit. We add one more digit to the right of x and obtain $10x$ more legal expressions of the length n.

(2) Let y be an arithmetic legal expression of $(n-2)$ symbols. Then we can add to the right of y one of the following 29 2-symbol expressions: $+0$, $+1$, $\ldots$, $+9$, $*0$, $*1$, $\ldots$, $*9$, $/1$, $\ldots$, $/9$ (no division by 0 is allowed).

We obtain the recurrence relation: $a_1 = 10$, $a_2 = 100$, $a_n = 10a_{n-1} + 29a_{n-2}$ for $n \geq 3$. We notice that $a_0 = 0$, indeed, $100 = a_2 = 10 \cdot a_1 + 29 \cdot a_0 = 10 \cdot 10 + 29 \cdot a_0$. i.e., $a_0 = 0$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 0$, $a_1 = 10$, $a_n = 10a_{n-1} + 29a_{n-2}$, $n \geq 2$.

Example. We would like to find a number of binary sequences of the length n without any consecutive 0's.

Let a_n denote the number of such sequences of length $n \geq 1$. Clearly, if $n = 1$, we have 0, 1, i.e., $a_1 = 2$, if $n = 2$, we have the sequences 01, 10, 11, i.e., $a_2 = 3$.

Let $n \geq 3$. Let $x_1 \cdots x_{n-2}x_{n-1}x_n$ be a sequence like that. There are two cases:

(1) The last symbol $x_n = 1$. Then the sequence $x_1 \cdots x_{n-2}x_{n-1}$ has no consecutive 0's.

(2) The last symbol $x_n = 0$. Then $x_{n-1} = 1$, and the sequence $x_1 \cdots x_{n-2}$ has no consecutive 0's.

Thus we conclude that $a_n = a_{n-1} + a_{n-2}$. Also we notice that the initial conditions $a_1 = 2$, $a_2 = 3$ could be replaced by $a_0 = 1$, $a_1 = 2$. Then $a_2 = a_1 + a_0 = 3$.

Exercise: Find a closed formula for the recurrence relation: $a_0 = 1$, $a_1 = 2$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.