SECOND MIDTERM EXAM REVIEW

(1) Prove that $n^5 - n$ is divisible by 5 for all integers $n \geq 1$.

(2) Prove that $2^{2n+1} + 1$ is divisible by 3 for all integers $n \geq 1$.

(3) Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all integers $n \geq 1$.

(4) Let n be a positive odd integer which is not divisible by 5. Prove that there exists an integer $\ell > 0$ such that the last digit of n^ℓ is equal to 1.

(5) Prove the inequalities: $2^n < \binom{2n}{n} < 4^n$ for all integers $n \geq 2$.

(6) Prove that $n^3 + 5n$ is divisible by 6 for all integers $n \geq 1$.

(7) Prove the identity:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(8) Prove the identity:

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

(9) Prove that $n^2 > n + 1$ for $n \geq 2$.

(10) Prove the identity:

$$r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all integers $n \geq 1$.

(11) How many seven-digit integers have the sum of their digits equal to 46?

(12) Prove that $8^{n+2} + 9^{2n+1}$ is divisible by 73 for all integers $n \geq 1$.

(13) Prove the inequalities: $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$ for all integers $n \geq 1$.

(14) How many *odd numbers* between 100,000 and 1,000,000 have no two digits the same?

(15) Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$.

- How many functions are there from the set A to the set B?
- How many of these functions map A onto B?
- How many ways are there to put 5 objects to 3 identical boxes so that no box will be left empty?

(16) How many integers between 1 and 10,000 (including 1 and 10,000) are not divisible by 2 or by 5 or by 17?

(17) How many positive integral solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_7 + x_8 + x_9 = 27\ ?$$

(18) Let n, k be positive integers.

- Let $n > k$. Prove the identity: $\displaystyle\sum_{r=0}^{n-1}(-1)^r \binom{n}{n-r}(n-r)^k = 0$.
- Prove the identity: $\displaystyle\sum_{r=0}^{n-1}(-1)^r \binom{n}{n-r}(n-r)^n = n!$.

(19) Let $A = \{a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

(a) How many functions are there from the set A to the set B?

(b) How many of these functions map A onto B?

(c) How many ways are there to put 9 objects to 4 identical boxes so that no box will be left empty?

(20) How many ways are there to put 21 objects in 5 boxes with at least 8 objects in one of the boxes?

(21) Prove that for any positive integer $n \geq 1$ the number $n^2 - 2$ is not divisible by 3.

(22) Let $x, y \in \mathbf{R}$.

(a) Prove that the statement $\forall x\ \exists y\ (x + y = 89)$ is true.

(b) Prove that the statement $\exists y\ \forall x\ \ (x + y = 89)$ is false.

(c) Prove that the statement $\forall y\ \forall x\ \ (x + y = 89)$ is false.

(d) Prove that the statement $\exists x\ \exists y\ (x + y = 89)$ is true.

(23) Show that the implication

$$\forall x\ (p(x) \vee q(x)) \to (\forall x\ p(x)) \vee (\forall x\ q(x))$$

is not a tautology.

(24) Show that the implication

$$(\exists x\ p(x)) \wedge (\exists x\ q(x)) \to \exists x\ (p(x) \wedge q(x))$$

is not a tautology.

(25) Negate the propositions:

(a) $\forall y\ \exists x\ [P(x) \to Q(y)]$

(b) $\exists y\ \exists x\ [P(x) \to Q(y)]$

(c) $\exists y\ \forall x\ [P(x) \to Q(y)]$

(d) $\exists y\ \exists x\ [P(x) \to Q(y)]$

(26) Compute the Euler function $\phi(n)$ for $n = 51$, $n = 452$, $n = 12,300$.

(29) For which positive integers n is $\phi(n)$ power of two? (Here $\phi(n)$ is the Euler function.)