Summary on Lecture 9, February 13, 2019

More on the principle of inclusion and exclusion.

Theorem. Let S be a finite set, and $c_1, \ldots, c_k$ be some conditions on elements of S. Then

$$N(\bar{c}_1 \cdots \bar{c}_k) = N + \sum_{\ell=1}^{k} (-1)^\ell \sum_{1 \leq i_1 < \cdots < i_\ell \leq k} N(c_{i_1} \cdots c_{i_\ell}),$$

where $N = |S|$, $N(c_{i_1} \cdots c_{i_\ell}) = |S_{i_1} \cap \cdots \cap S_{i_\ell}|$, and $N(\bar{c}_1 \cdots \bar{c}_k) = |\overline{S_1 \cup \cdots \cup S_k}|$.

Important Examples. (1) Let $A = \{1, 2, \ldots, 999, 999\}$. Count how many elements $n \in A$ have the property that a sum of digits of n is equal to 35?

Solution. Let $x_1, \ldots, x_6$ denote digits of $n = x_1x_2x_3x_4x_5x_6$. Then the condition on n is equivalent to the following question. Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 35$, and the integers x_i are such that $0 \leq x_i \leq 9$, $i = 1, \ldots, 6$. How many integral solutions (i.e. when all x_i are integers) are there?

First, we consider all solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 35$ such that $0 \leq x_i$, $i = 1, \ldots, 6$. We denote the set of all such solutions by S. For each $i = 1, 2, 3, 4, 5, 6$, we say that a solution $x_1 \ldots x_6$ satisfies the property c_i if $x_i \geq 10$. We denote by S_i the set of solutions satisfying c_i. Then we compute:

$$\begin{aligned}
N &= |\mathsf{S}| = \binom{35+6-1}{6-1} = \binom{40}{5} \\
N(c_i) &= |\mathsf{S}_i| = \binom{25+6-1}{6-1} = \binom{30}{5} \\
N(c_{i_1}c_{i_2}) &= |\mathsf{S}_{i_1} \cap \mathsf{S}_{i_2}| = \binom{15+6-1}{6-1} = \binom{20}{5} \\
N(c_{i_1}c_{i_2}c_{i_3}) &= |\mathsf{S}_{i_1} \cap \mathsf{S}_{i_2} \cap \mathsf{S}_{i_3}| = \binom{5+6-1}{6-1} = \binom{10}{5} \\
N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}) &= N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}c_{i_5}) = N(c_1c_2c_3c_4c_5c_6) = 0
\end{aligned}$$

Then we compute the answer:

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6) = \binom{40}{5} - \binom{6}{1} \cdot \binom{30}{5} + \binom{6}{2} \cdot \binom{20}{5} - \binom{6}{3} \cdot \binom{10}{5}$$

(2) Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$. A function $f : A \to B$ is a rule which for each element $a_i \in A$ assigns an element $f(a_i) \in B$. Let $\mathcal{F}(A, B)$ be the set of all functions $f : A \to B$.

Exercise. Prove that $|\mathcal{F}(A, B)| = n^m$.

Definition. Let $f : A \to B$ be a function. We denote by $f(A) = \{\, f(a) \mid a \in A\} \subset B$ the image of f. We say that a function $f : A \to B$ is *onto* iff $f(A) = B$. Let $\mathcal{F}^{\text{onto}}(A, B) \subset \mathcal{F}(A, B)$ be the set of all functions $f : A \to B$ which are onto.

Question: Let $|A| = m$ and $|B| = n$. What is the size of the set $\mathcal{F}^{\text{onto}}(A, B)$?

Solution. We denote $\mathcal{F} := \mathcal{F}(A, B)$. Then we say that a function $f : A \to B$ satisfies c_i iff $b_i \notin f(A)$, where $i = 1, \ldots, n$. We denote by $\mathcal{F}_i$ the set of all functions satisfying c_i. Then we have:

$$\begin{array}{rcl} N & = & |\mathcal{F}| = n^m \\ N(c_i) & = & |\mathcal{F}_i| = (n-1)^m \\ N(c_{i_1}c_{i_2}) & = & |\mathcal{F}_{i_1} \cap \mathcal{F}_{i_2}| = (n-2)^m \\ N(c_{i_1}c_{i_2}c_{i_3}) & = & |\mathcal{F}_{i_1} \cap \mathcal{F}_{i_2} \cap \mathcal{F}_{i_3}| = (n-3)^m \\ \cdots & & \cdots \\ N(c_{i_1}\cdots c_{i_k}) & = & |\mathcal{F}_{i_1} \cap \cdots \cap \mathcal{F}_{i_k}| = (n-k)^m \\ \cdots & & \cdots \\ N(c_{i_1}\cdots c_{i_{n-1}}) & = & |\mathcal{F}_{i_1} \cap \cdots \cap \mathcal{F}_{i_{n-1}}| = 1^m \\ N(c_{i_1}\cdots c_{i_n}) & = & |\mathcal{F}_{i_1} \cap \cdots \cap \mathcal{F}_{i_n}| = 0 \end{array}$$

We obtain the answer:

$$|\mathcal{F}^{\text{onto}}(A,B)| = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots (-1)^k \binom{n}{k}(n-k)^m + \cdots (-1)^{n-1}\binom{n}{n-1}1^m$$

(3) Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$. Then $N = |\mathcal{F}(A,B)| = 5^6$,

$$\begin{array}{lcl} N & = & 5^6 \\ N(c_i) & = & 4^6 \\ N(c_{i_1}c_{i_2}) & = & 3^6 \\ N(c_{i_1}c_{i_2}c_{i_3}) & = & 2^6 \\ N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}) & = & 1^6 \end{array}$$

We obtain the answer:

$$\begin{array}{rcl} |\mathcal{F}^{\text{onto}}(A,B)| & = & 5^6 - \binom{5}{1}4^6 + \binom{5}{2}3^6 - \binom{5}{3}2^6 + \binom{5}{4}1^6 \\ & = & 15,625 - 5 \cdot 4,096 + 10 \cdot 729 - 10 \cdot 64 + 5 \cdot 1 \\ & = & 15,625 - 20,480 + 7,290 - 640 + 5 = 1,800 \end{array}$$

(4) **Euler function.** For given positive integer n, consider the set of numbers m such that $1 \le m < n$ and $\gcd(m,n) = 1$. Leonard Euler defined the function:

$$\phi(n) = |\{\ m \mid 1 \le m < n, \text{ and } \gcd(m,n) = 1\ \}|.$$

Here is the values of $\phi(n)$ for some n:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\phi(n)$	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16

There is a simple formula to compute $\phi(n)$. Recall that for every integer n there exist primes $p_1, \ldots, p_s$ and positive $e_1, \ldots, e_s$ such that $n = p_1^{e_1} \cdots p_s^{e_s}$. Here is the formula:

$$\phi(n) = n \prod_{i=1}^{s} \left(1 - \frac{1}{p_i}\right)$$

Example: Let $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4}$, and $S = \{1, \ldots, n\}$.

We notice that for $m < n$ with $\gcd(m,n) > 1$, m has to be divisible by one of the primes p_i. We say that "m satifies c_i" iff $p_i | m$. Let

$$S_i = \{ \ m \in S \mid p_i | m \ \}, \ i = 1,2,3,4.$$

Then $N = |S| = n$, $N(c_i) = |S_i| = \frac{n}{p_i}$. Then $N(c_i c_j) = \frac{n}{p_i p_j}$, $N(c_i c_j c_k) = \frac{n}{p_i p_j p_k}$, $N(c_1 c_2 c_3 c_4) = \frac{n}{p_1 p_2 p_3 p_4}$. Then

$$\begin{aligned} N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) &= n - \left(\tfrac{n}{p_1} + \tfrac{n}{p_2} + \tfrac{n}{p_3} + \tfrac{n}{p_4} \right) + \left(\tfrac{n}{p_1 p_2} + \tfrac{n}{p_1 p_3} + \tfrac{n}{p_1 p_4} + \tfrac{n}{p_2 p_3} + \tfrac{n}{p_2 p_4} + \tfrac{n}{p_2 p_4} \right) \\ & \quad - \left(\tfrac{n}{p_1 p_2 p_3} + \tfrac{n}{p_1 p_2 p_4} + \tfrac{n}{p_1 p_3 p_4} + \tfrac{n}{p_2 p_3 p_4} \right) + \tfrac{n}{p_1 p_2 p_3 p_4} \end{aligned}$$

It is easy to check:

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = \tfrac{n(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)}{p_1 p_2 p_3 p_4} = n\left(1 - \tfrac{1}{p_1}\right)\left(1 - \tfrac{1}{p_2}\right)\left(1 - \tfrac{1}{p_3}\right)\left(1 - \tfrac{1}{p_4}\right).$$

Examples: (1) Let p be a prime. Then $\phi(p) = p - 1$, and $\phi(p^k) = p^{k-1}(p-1)$.
(2) Since $2019 = 3 \cdot 673$, where 673 is a prime number. We obtain:

$$\phi(2019) = 3 \cdot 673 \left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{673}\right) = 2 \cdot 672 = 1342.$$

Recursive definitions. There are many mathematical objects which we can define only *recursively*. We start with well-known example:

(1) **Fibonacci numbers** F_n. We define:

(B) $F_0 = 0$, $F_1 = 1$,

(R) $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Here are the first few values of F_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

We prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ by induction. Indeed, it's true if $n = 1$.

Assume $\sum_{i=1}^{k} F_i^2 = F_k F_{k+1}$. Then

$$\begin{aligned} \sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^{k} F_i^2 + F_{k+1}^2 \\ &= F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}. \end{aligned}$$

(2) We define a sequence of numbers a_n as:

(B) $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and

(R) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

Here are the first few values of a_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	0	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377

We notice that $a_n = F_{n-1}$ for $n \geq 3$. We would like to prove that $a_{n+3} \geq (\sqrt{2})^n$ for all $n \geq 0$. Indeed, it's true if $n = 0, 1$. Assume $a_{k+3} \geq (\sqrt{2})^k$ for all $k = 0, 1, \ldots, n$. We should prove that $a_{n+4} \geq (\sqrt{2})^{n+1}$. We have:

$$\begin{aligned} a_{n+4} = a_{n+3} + a_{n+2} &\geq (\sqrt{2})^n + (\sqrt{2})^{n-1} \\ &= (\sqrt{2})^{n-1}(\sqrt{2}+1) \geq (\sqrt{2})^{n-1} \cdot 2 = (\sqrt{2})^{n+1}. \end{aligned}$$

Here we use that $\sqrt{2} + 1 \geq 2$ and $2 = (\sqrt{2})^2$.

(3) We can define recursively the binomial coefficients $\binom{n}{r}$:

(B) $\binom{n}{0} = 1$, $\binom{n}{r} = 0$ if $r < 0$ and $r > n$.

(R) $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$.

(4) We define factorial $\texttt{FAC}(n)$:

(B) $\texttt{FAC}(0) = 1$

(R) $\texttt{FAC}(n) = \texttt{FAC}(n-1) \cdot n$ for $n \geq 1$.

(5) We define the Harmonic numbers H_n:

(B) $H_1 = 1$

(R) $H_n = H_{n-1} + \frac{1}{n}$ for $n \geq 2$.

(6) We define the sequence $\texttt{SEC}(n)$:

(B) $\texttt{SEC}(0) = 1$

(R) $\texttt{SEC}(n+1) = \frac{n+1}{\texttt{SEC}(0)}$.

Exercise. Use induction to prove that the sequence $\texttt{SEC}(n)$ is well-defined.

(7) We define the sequence $T(n)$ as follows:

(B) $T(1) = 1$

(R) $T(n) = 2 \cdot T(\lfloor \frac{n}{2} \rfloor)$ for $n \geq 2$.

We compute a couple of values of $T(n)$:

$$\begin{aligned} T(73) &= 2 \cdot T(36) = 2^2 \cdot T(18) = 2^3 \cdot T(9) = 2^4 \cdot T(4) = 2^5 \cdot T(2) = 2^6 \\ T(2019) &= 2 \cdot T(1009) = 2^2 \cdot T(504) = 2^3 \cdot T(252) = 2^4 \cdot T(126) = 2^5 \cdot T(63) \\ &= 2^6 \cdot T(31) = 2^7 \cdot T(15) = 2^8 \cdot T(7) = 2^9 \cdot T(3) = 2^{10} \end{aligned}$$

Exercise. Use induction to prove that $T(n) = \max\{\ 2^k \mid 2^k \leq n\ \}$.

Exercise. Define a sequence $S(n)$ such that $S(n) = \min\{\ 2^k \mid n \leq 2^k\ \}$.

Exercise. Let p be a prime. Define recursively a sequence $T_p(n)$ such that

$$T(n) = \max\{\ p^k \mid p^k \leq n\ \}.$$

Exercise. Let p be a prime. Define recursively a sequence $S_p(n)$ such that

$$S_p(n) = \min\{\ p^k \mid n \leq p^k\ \}.$$

Exercise. Define recursively what does it mean "well-formed formula", see Ex. 17, p. 220.