Summary on Lecture 8, February 11, 2019

- **Examples on induction:**

(1) Prove that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$.

(2) Prove that $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

(3) Prove that $\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

(4) Prove that $8^n - 2^n$ is divisible by 6 for every $n \in \mathbf{Z}_+$.

(5) Prove that $11^n - 4^n$ is divisible by 7 for every $n \in \mathbf{Z}_+$.

(6) Prove that $8^{n+2} + 9^{2n+1}$ is divisible by 73 for every $n \in \mathbf{Z}_+$.

(7) Prove that $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for every $n \in \mathbf{Z}_+$.
Proof. Let $n = 1$. Then $5^{1+1} + 2 \cdot 3^1 + 1 = 32$. OK
Assume that $5^{k+1} + 2 \cdot 3^k + 1 = 8 \cdot \ell$. Consider the case $n = k + 1$.

$$
\begin{aligned}
5^{k+2} + 2 \cdot 3^{k+1} + 1 &= 5 \cdot 5^{k+1} + 5 \cdot 2 \cdot 3^k + 5 \cdot 1 - 5 \cdot 2 \cdot 3^k - 5 \cdot 1 + 2 \cdot 3^{k+1} + 1 \\
&= 5 \cdot (5^{k+1} + 2 \cdot 3^k + 1) - 10 \cdot 3^k + 6 \cdot 3^k - 4 \\
&= 5 \cdot (5^{k+1} + 2 \cdot 3^k + 1) - 4(3^k + 1).
\end{aligned}
$$

By induction, $5 \cdot (5^{k+1} + 2 \cdot 3^k + 1)$ is divisible by 8, then $(3^k + 1)$ is always even. Hence $4(3^k + 1)$ is divisible by 8. We obtain that $5^{k+2} + 2 \cdot 3^{k+1} + 1$ is divisible by 8.

(8) We define the *harmonic numbers:* $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3}$, and $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $H_{2^n} \geq 1 + \frac{n}{2}$.
Proof. Let $n = 1$, then $H_2 = 1 + \frac{1}{2}$. By induction, we have $H_{2^k} \geq 1 + \frac{k}{2}$. Then we have:

$$
\begin{aligned}
H_{2^{k+1}} &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{2^k} + \tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^k+2^k} \\
&= H_{2^k} + \tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^k+2^k}.
\end{aligned}
$$

We notice that $2^k + i \leq 2^k + 2^k$ for each $i = 1, 2, \ldots, 2^k$. Then we have that $\frac{1}{2^k+i} \geq \frac{1}{2^k+2^k}$.
Then we have:

$$
\begin{aligned}
H_{2^{k+1}} &= H_{2^k} + \tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \cdots + \tfrac{1}{2^k+2^k}. \\
&\geq H_{2^k} + \tfrac{2^k}{2^k+2^k} \\
&= H_{2^k} + \tfrac{2^k}{2 \cdot 2^k} \\
&= H_{2^k} + \tfrac{1}{2}
\end{aligned}
$$

By induction, $H_{2^k} \geq 1 + \frac{k}{2}$. Then we have:

$$H_{2^{k+1}} \geq H_{2^k} + \frac{1}{2} \geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}.$$

Remark.[1] In particular, it means that $\lim_{n \to \infty} H_n = \infty$.

(9) Prove that $n^2 > n + 1$ for all $n \geq 2$.

(10) Prove the following inequalities for all $n \in \mathbf{Z}_+$:

$$\sqrt{n} \leq \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1.$$

- **The principle of inclusion and exclusion.** We start with an example. Let $S = \{1, 2, \ldots, 10,000\}$. We choose two primes $p_1 = 7$ and $p_2 = 11$, and we let

$$\begin{array}{rcl} S_1 & := & \{\ n \in S \mid n \text{ is divisible by } p_1 = 7\ \}, \\ S_2 & := & \{\ n \in S \mid n \text{ is divisible by } p_2 = 11\ \}. \end{array}$$

We say that "$n \in S$ satisfies a property c_1" iff $n \in S_1$, and "$n \in S$ satisfies a property c_2" iff $n \in S_2$. We use the notations: $N(c_1) = |S_1|$, $N(c_2) = |S_2|$, $N(c_1c_2) = |S_1 \cap S_2|$. Then in these terms, we have:

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| = N(c_1) + N(c_2) - N(c_1c_2).$$

Then we denote:

$$N(\bar{c}_1c_2) = |\bar{S}_1 \cap S_2| = |\{n \in S \mid n \text{ does not satisfy } c_1 \text{ and satisfies } c_2\}|,$$

$$N(c_1\bar{c}_2) = |S_1 \cap \bar{S}_2| = |\{n \in S \mid n \text{ satisfies } c_1 \text{ and does not satisfy } c_2\}|,$$

$$N(\bar{c}_1\bar{c}_2) = |\bar{S}_1 \cap \bar{S}_2| = |\{n \in S \mid n \text{ does not satisfy } c_1 \text{ and does not satisfy } c_2\}|.$$

We compute: $N(c_1) = \lfloor \frac{10,000}{7} \rfloor = 1,428$, $N(c_2) = \lfloor \frac{10,000}{11} \rfloor = 909$, $N(c_1c_2) = \lfloor \frac{10,000}{77} \rfloor = 129$. Then:

$$\begin{array}{rcl} N(\bar{c}_1c_2) & = & N(c_2) - N(c_1c_2) = 909 - 129 = 780, \\ N(c_1\bar{c}_2) & = & N(c_1) - N(c_1c_2) = 1,428 - 129 = 1,299, \\ N(\bar{c}_1\bar{c}_2) & = & N - [N(c_1) + N(c_2) - N(c_1c_2)] \\ & = & 10,000 - [1,428 + 909 - 129] = 7,792. \end{array}$$

We add one more condition: $n \in S$ satisfies c_3 iff n is divisible by 23. Then we compute $N(\bar{c}_1\bar{c}_2\bar{c}_3)$:

$$\begin{array}{rcl} N(\bar{c}_1\bar{c}_2\bar{c}_3) & = & |\bar{S}_1 \cap \bar{S}_2 \cap \bar{S}_3| = |\overline{S_1 \cup S_2 \cup S_3}| \\ & = & N - [(N(c_1) + N(c_2) + N(c_3)) - ((N(c_1c_2) + N(c_1c_3) + N(c_2c_3)) + N(c_1c_2c_3)] \\ & = & 7,456 \end{array}$$

[1] For those who are good friends with calculus

Exercise. Verify this calculation.

Theorem. Let S be a finite set, and $c_1, \dots, c_k$ be some conditions on elements of S. Then

$$N(\bar{c}_1 \cdots \bar{c}_k) = N + \sum_{\ell=1}^{k} (-1)^\ell \sum_{1 \le i_1 < \cdots < i_\ell \le k} N(c_{i_1} \cdots c_{i_\ell}),$$

where $N = |S|$, $N(c_{i_1} \cdots c_{i_\ell}) = |S_{i_1} \cap \cdots \cap S_{i_\ell}|$, and $N(\bar{c}_1 \cdots \bar{c}_k) = |\overline{S_1 \cup \cdots \cup S_k}|$.

Important Examples. (1) Let $A = \{1, 2, \dots, 999, 999\}$. Count how many elements $n \in A$ have the property that a sum of digits of n is equal to 35?

Solution. Let $x_1, \dots, x_6$ denote digits of $n = x_1 \dots x_6$. Then the condition on n is equivalent to the following question. Consider the equation $x_1 + x_2 + xc_3 + x_4 + x_5 + x_6 = 35$, and the integers x_i are such that $0 \le x_i \le 9$, $i = 1, \dots, 6$. How many integral solutions (i.e. when all x_i are integers) are there?

First, we consider all solutions of the equation $x_1 + x_2 + xc_3 + x_4 + x_5 + x_6 = 35$ such that $0 \le x_i$, $i = 1, \dots, 6$. We denote the set of all such solutions by S. For each $i = 1, 2, 3, 4, 5, 6$, we say that a solution $x_1 \dots x_6$ satisfies the property c_i if $x_i \ge 10$. We denote by S_i the set of solutions satisfying c_i. Then we compute:

$$\begin{aligned}
N &= |\mathsf{S}| = \binom{35+6-1}{6-1} = \binom{40}{5} \\
N(c_i) &= |\mathsf{S}_i| = \binom{25+6-1}{6-1} = \binom{30}{5} \\
N(c_{i_1} c_{i_2}) &= |\mathsf{S}_{i_1} \cap \mathsf{S}_{i_2}| = \binom{15+6-1}{6-1} = \binom{20}{5} \\
N(c_{i_1} c_{i_2} c_{i_3}) &= |\mathsf{S}_{i_1} \cap \mathsf{S}_{i_2} \cap \mathsf{S}_{i_3}| = \binom{5+6-1}{6-1} = \binom{10}{5} \\
N(c_{i_1} c_{i_2} c_{i_3} c_{i_4}) &= N(c_{i_1} c_{i_2} c_{i_3} c_{i_4} c_{i_5}) = N(c_1 c_2 c_3 c_4 c_5 c_6) = 0
\end{aligned}$$

Then we compute the answer:

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4 \bar{c}_5 \bar{c}_6) = \binom{40}{5} - \binom{6}{1} \cdot \binom{30}{5} + \binom{6}{2} \cdot \binom{20}{5} - \binom{6}{3} \cdot \binom{10}{5}$$

(2) Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_n\}$. A function $f : A \to B$ is a rule which for each element $a_i \in A$ assigns an element $f(a_i) \in B$. Let $\mathcal{F}(A, B)$ be the set of all functions $f : A \to B$.
Exercise. Prove that $|\mathcal{F}(A, B)| = n^m$.
Let $f : A \to B$ be a function. We denote by $f(A) = \{ f(a) \ |a \in A\} \subset B$ the image of f. We say that a function $f : A \to B$ is *onto* iff $f(A) = B$. Let $\mathcal{F}^{\text{onto}}(A, B) \subset \mathcal{F}(A, B)$ be the set of all functions $f : A \to B$ which are onto.
Question: What is the size of the set $\mathcal{F}^{\text{onto}}(A, B)$?
Solution. We denote $\mathcal{F} := \mathcal{F}(A, B)$. Then we say that a function $f : A \to B$ satisfies c_i iff

$b_i \notin f(A)$, where $i = 1, \ldots, n$. We denote by $\mathcal{F}_i$ the set of all functions satisfying c_i. Then we have:

$$\begin{aligned}
N &= |\mathcal{F}| = n^m \\
N(c_i) &= |\mathcal{F}_i| = (n-1)^m \\
N(c_{i_1}c_{i_2}) &= |\mathcal{F}_{i_1} \cap \mathcal{F}_{i_2}| = (n-2)^m \\
N(c_{i_1}c_{i_2}c_{i_3}) &= |\mathcal{F}_{i_1} \cap \mathcal{F}_{i_2} \cap \mathcal{F}_{i_3}| = (n-3)^m \\
N(c_{i_1} \cdots c_{i_k}) &= |\mathcal{F}_{i_1} \cap \cdots \cap \mathcal{F}_{i_k}| = (n-k)^m
\end{aligned}$$

We obtain the answer:

$$|\mathcal{F}^{\text{onto}}(A, B)| = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots (-1)^k \binom{n}{k}(n-k)^m + \cdots (-1)^{n-1}\binom{n}{n-1} 1^m$$