Summary on Lecture 6, February 4, 2019

- **Two proofs.**

(1) Show that n^2-2 is not divisible by 5 for every $n \in \mathbf{Z}_+$.
Proof. We consider the cases: (a) $n=5k$, (b) $n=5k+1$, (c) $n=5k+2$, (d) $n=5k+3$, (e) $n=5k+4$.
(a): We have $n^2-2=25k^2-2$ is not divisible by 5.
(b): We have $n^2-2=25k^2+10k+1-2=5(5k^2+2k)-1$ is not divisible by 5.
(c): We have $n^2-2=25k^2+20k+4-2=5(5k^2+4k)+2$ is not divisible by 5.
(d): We have $n^2-2=25k^2+30k+9-2=5(5k^2+6k+1)+2$ is not divisible by 5.
(e): We have $n^2-2=25k^2+40k+16-2=5(5k^2+8k+3)+1$ is not divisible by 5.
Thus n^2-2 is not divisible by 5 for every $n \in \mathbf{Z}_+$.

(2) Show that n^4-n^2 is divisible by 3 for every $n \in \mathbf{Z}_+$.
Proof. We notice that $n^4-n^2=n^2(n^2-1)=n^2(n-1)(n+1)$. Then we consider the cases: (a) $n=3k$, (b) $n=3k+1$, (c) $n=3k+2$.
(a): Then n^2 is divisible by 3.
(b): Then $n-1$ is divisible by 3.
(c): Then $n+1$ is divisible by 3.
Thus n^4-n^2 is divisible by 3 for every $n \in \mathbf{Z}_+$.

(3) Show that n^4-n^2 is even for every $n \in \mathbf{Z}_+$.

(4) Show that n^4-n^2 is divisible by 6 for every $n \in \mathbf{Z}_+$.

- **Quantifiers.** We introduce two important notations:

"$\forall$" "for all"

"$\exists$" "exists"

(a) Let $p(n)$ means "$n^2=n$", where $n \in \mathbf{Z}$. Then we have the statements:
$\forall n\ p(n) \iff \forall n\ (n^2=n)$ F
$\exists n\ p(n) \iff \exists n\ (n^2=n)$ T

(b) Let $p(n)$ means "$n+2$ is even", where $n \in \mathbf{Z}$. Then we have the statements:
$\forall n\ p(n) \iff \forall n\ (n+2 \text{ is even})$ F
$\exists n\ p(n) \iff \exists n\ (n+2 \text{ is even})$ T
$\forall n\ \neg p(n) \iff \forall n\ \neg(n+2 \text{ is even}) \iff \forall n\ (n+2 \text{ is odd})$ F
$\exists n\ \neg p(n) \iff \exists n\ \neg(n+2 \text{ is even}) \iff \exists n\ (n+2 \text{ is odd})$ T

We notice the following tautologies:

$$\neg(\exists x\ p(x)) \iff \forall x\ \neg p(x)$$

$$\neg(\forall x\ p(x)) \iff \exists x\ \neg p(x)$$

More examples:

(a) Let $x, y \in \mathbf{R}$.
$\forall x\ \forall y\ (x+y=y+x)$ T

(b) Let $n \in \mathbf{Z}$.
$\forall n$ [(n is a prime) $\to$ (n is a odd)] F
$\forall n$ [((n is a prime) $\wedge$ $(n \geq 3)$) $\to$ (n is a odd)] T

(c) Let $n \in \mathbf{Z}$.
$\forall n \ (n + 6 \leq 2^n)$ F
$\forall n \ [(n + 6 \leq 2^n) \wedge (n \geq 4)]$ T
Here $n = 3$ is a counterexample for the first statement. The second one is hard to prove (we'll learn it soon: **induction**).

(d) Let $x, y \in \mathbf{R}$.
$\forall x \ \forall y \ [(x > y) \to (x^2 > y^2)]$ F
A counterexample: $x = 1$, $y = -2$.

(e) Let $x, y \in \mathbf{R}_+ = \{ \ r \in \mathbf{R} \mid r > 0 \ \}$.
$\forall x \ \forall y \ [(x > y) \to (x^2 > y^2)]$ T

(f) Let $x, y \in \mathbf{R}$, and $p(x) := (x \geq 0)$, $q(x) := (x^2 \geq 0)$.
$\exists x \ (p(x) \to q(x))$ T
$\forall x \ (p(x) \to q(x))$ T

(g) We notice that the implication $\forall x \ p(x) \to \exists x \ p(x)$ is a tautology.

(h) Let $x \in \mathbf{Z}_+$.
$\forall x \ (x^2 \geq 1)$ T
Let $x \in \mathbf{Z}$.
$\forall x \ (x^2 \geq 1)$ F

(i) Important tautology: $\forall x \ (p(x) \to q(x)) \Longleftrightarrow \forall x \ (\neg q(x) \to \neg p(x))$

(j) More tautologies:

$$\exists x \ (p(x) \wedge q(x)) \Longrightarrow (\exists x \ p(x)) \wedge (\exists x \ q(x))$$

We notice that the implication

$$(\exists x \ p(x)) \wedge (\exists x \ q(x)) \to \exists x \ (p(x) \wedge q(x))$$

is not a tautology. Example: $p(x) = (x < 1)$, $q(x) = (x \geq 1)$. Then the statement $(\exists x \ p(x)) \wedge (\exists x \ q(x))$ is true, but the statement $\exists x \ (p(x) \wedge q(x))$ is false. More tautologies:

$$\exists x \ (p(x) \vee q(x)) \Longleftrightarrow (\exists x \ p(x)) \vee (\exists x \ q(x))$$

$$\forall x \ (p(x) \wedge q(x)) \Longleftrightarrow (\forall x \ p(x)) \wedge (\forall x \ q(x))$$

$$(\forall x \ p(x)) \vee (\forall x \ q(x)) \Longrightarrow \forall x \ (p(x) \vee q(x))$$

We notice that the implication

$$\forall x \ (p(x) \vee q(x)) \to (\forall x \ p(x)) \vee (\forall x \ q(x))$$

is not a tautology. The same example: $p(x) = (x < 1)$, $q(x) = (x \geq 1)$. Then the statement $\forall x \ (p(x) \vee q(x))$ is true, but the statement $(\forall x \ p(x)) \vee (\forall x \ q(x))$ is false.

(k) Let $x, y \in \mathbf{R}$. Then the statement $\forall x \ \exists y \ (x + y = 25)$ is true. Indeed for any given $x = a$, we can find $y = 25 - a$ so that $x + y = 25$.

(l) However the statement $\exists y\ \forall x\ \ (x+y=25)$ is false. Indeed assume that there exists $y=b$ so that for every x we have $x+b=25$. Then this is true for $x=25-b$, but not for all x.

(m) Check that the statement $\forall y\ \forall x\ \ (x+y=25)$ is false, and the statement $\exists x\ \exists y\ (x+y=25)$ is true.

Limits. Next we discuss definitions of $\lim_{n\to\infty} x_n$ and $\lim_{x\to a} f(x)$.

- Let $\{x_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} x_n = A$ if and only if for every $\epsilon > 0$ there exists an integer N such that for every n $(n > N)$ implies that $|x_n - A| < \epsilon$. In our terms, the following proposition

$$\forall\epsilon > 0\ \exists N\ \forall n[(n > N) \to (|x_n - A| < \epsilon)]$$

is true. What does it mean that $\lim_{n\to\infty} x_n \neq A$? The answer:

$$\neg(\forall\epsilon > 0\ \exists N\ \forall n[(n > N) \to (|x_n - A| < \epsilon)]) \iff \exists\epsilon > 0\ \forall N\ \exists n\ \neg[(n > N) \to (|x_n - A| < \epsilon)]$$

$$\iff \exists\epsilon > 0\ \forall N\ \exists n[(n > N) \wedge (|x_n - A| \geq \epsilon)].$$

- Let $f(x)$ be a function. We say that $\lim_{x\to a} f(x) = L$ is for every $\epsilon > 0$ there exists $\delta > 0$ such that for every x the inequality $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$. In our terms, the following proposition

$$\forall\epsilon > 0\ \exists\delta > 0\ \forall x[(0 < |x-a| \to (|f(x) - L| < \epsilon)]$$

is true. What does it mean that $\lim_{x\to a} f(x) \neq L$? The answer:

$$\neg\{\forall\epsilon > 0\ \exists\delta > 0\ \forall x[(0 < |x-a| \to (|f(x) - L| < \epsilon)]\} \iff$$

$$\exists\epsilon > 0\ \forall\delta > 0\ \exists x\neg[(0 < |x-a| \to (|f(x) - L| < \epsilon)] \iff$$

$$\exists\epsilon > 0\ \forall\delta > 0\ \exists x[(0 < |x-a| \wedge (|f(x) - L| \geq \epsilon)].$$

Those two examples are very important to understand really well.

- Give examples when $\lim_{n\to\infty} x_n \neq A$ and $\lim_{x\to a} f(x) \neq L$.