Summary on Lecture 4, January 16, 2019

(1) **Propositions.** A proposition will be any sentence that is either true or false, but not both. The following examples are propositions:

(a) Napoleon lived in the 20th century.
(b) $2 \times 7 = 14$.
(c) $2 + 3 = 7$.
(d) The number 4 is positive and the number 3 is negative.
(e) If a set has n elements, then it has 2^n subsets.
(f) $2^n + n$ is a prime number for infinitely many n.
(g) Every even integer greater than 2 is the sum of two prime numbers.
(h) If the Earth is flat, then $2 + 3 = 4$.

The following sentences are not propositions:

(a) Mr. P. is a bad president.
(b) $x - y = y - x$.
(c) $A^2 = 0$ implies $A = 0$.
(d) Math is fun.
(e) What a beautiful sunset!
(f) Get up and work.

(2) If p, q are propositions, we can form new "compound propositions":

$\neg p$: "not p"
$p \vee q$: "p or q"
$p \wedge q$: "p and q"
$p \rightarrow q$: "p implies q" or "p is sufficient for q", or "q is necessary for p".
$p \leftrightarrow q$: "p is and only if q" or "p is sufficient and necessary for q".

The propositions $p \rightarrow q$, $q \rightarrow p$, $\neg q \rightarrow \neg p$ appear to be related and are sometimes confused with each other. It is important to keep them straight. The proposition $q \rightarrow p$ is called the **converse** of the proposition $p \rightarrow q$. As we will see, it has a different meaning from $p \rightarrow q$. It turns out that $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$, which is called the **contrapositive** of $p \rightarrow q$.

Consider the sentence *"If it is raining, then there are clouds in the sky"*. This is the compound proposition $p \rightarrow q$, where $p =$ *"it is raining"* and $q =$ *"there are clouds in the sky"* (true).

The converse of $p \rightarrow q$ reads: *"If there are clouds in the sky then it is raining"* (false).

The contrapositive of $p \rightarrow q$ reads: *"If there are no clouds in the sky, then it is not raining."* (true).

Examples:

(1) If -1 is a positive number, then $2+2=5$. **True:** the hypothesis is obviously false, thus no matter what the conclusion, the implication holds

(2) If -1 is a positive number, then $2+2=4$ **True:** for the same reason as above.

(3) If $\sin x = 0$ then $x=0$. **False:** If $x=2\pi$, then $\sin\pi = 0$, but $\pi \neq 0$.

Basic true-false tables:

p	$\neg p$
0	1
1	0

p	q	$p \vee q$
0	0	0
1	0	1
0	1	1
1	1	1

p	q	$p \wedge q$
0	0	0
1	0	0
0	1	0
1	1	1

p	q	$p \to q$
0	0	1
1	0	0
0	1	1
1	1	1

p	q	$p \leftrightarrow q$
0	0	1
1	0	0
0	1	0
1	1	1

Two compound propositions s_1 and s_2 are equivalent if s_1 is true if and only if s_1 is true. Then we write $s_1 \Longleftrightarrow s_2$.

Exercises:

(1) Show that $p \to q$ and $\neg q \to \neg p$ are equivalent.

(2) Show that $p \to q$ and $\neg p \vee q$ are equivalent.

(3) Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are equivalent.

(4) Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equivalent.

We denote by $\mathbf{T}_0$ a *tautology*, i.e. a proposition which is always true, and by $\mathbf{F}_0$ a *contradiction*, i.e. a proposition which is always false.

Examples to analyze:

(1) $(p \vee \neg p) \Longleftrightarrow \mathbf{T}_0$;

(2) $(p \wedge \neg p) \Longleftrightarrow \mathbf{F}_0$;

(3) $(p \vee \mathbf{F}_0) \Longleftrightarrow p$;

(4) $(p \vee \mathbf{T}_0) \Longleftrightarrow \mathbf{T}_0$;

(5) $(p \wedge \mathbf{F}_0) \Longleftrightarrow \mathbf{F}_0$;

(6) $(p \wedge \mathbf{T}_0) \Longleftrightarrow p$;

More examples to analyze:

(7) $[(p \to r) \wedge (q \to r)] \Longleftrightarrow [(p \vee q) \to r]$;

(8) $[(p \to q) \wedge (p \to r)] \Longleftrightarrow [p \to (q \wedge r)]$;

(9) $[(p \wedge q) \to r] \Longleftrightarrow [p \to (q \to r)]$;

(10) $(p \to q) \Longleftrightarrow [(p \wedge \neg q) \to \mathbf{F}_0]$.