Summary on Lecture 3, January 14, 2019

First, we return to the exercises at the end of the Lecture 2.

Exercises:

(1) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the equation $x_1 + \cdots + x_n = r$.

(2) Determine number of integral solutions $x_i \geq 1$, $i = 1, \ldots, n$ of the equation $x_1 + \cdots + x_n = r$.

(3) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the inequality $x_1 + \cdots + x_n \leq r$.

(4) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the inequality $x_1 + \cdots + x_n < r$.

(5) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 9?

(6) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 10?

(7) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 14?

(8) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 21?

Next, we consider several different examples related to the previous ones:

(9) We determine all different ways we can decompose an integer into a sum of non-zero integers.

Example $n = 4$:

$$\begin{array}{cccc} & 4=1+3 & 4=1+1+2 & \\ 4=4 & 4=3+1 & 4=1+2+1 & 4=1+1+1+1 \\ & 4=2+2 & 4=2+1+1 & \end{array}$$

Totally we have $8 = 2^{4-1}$ different ways. In general, for given n, we have the cases:

1 summand	$n = x_1$	$x_1 \geq 1$	$n-1 = y_1,$	$y_1 \geq 0,$	1
2 summands	$n = x_1 + x_2$	$x_1, x_2 \geq 1$	$n-2 = y_1 + y_2$	$y_1, y_2 \geq 0$	$\binom{n-1}{1}$
3 summands	$n = x_1 + x_2 + x_3$	$x_1, x_2, x_3 \geq 1$	$n-3 = y_1 + y_2 + y_3$	$y_1, y_2, y_3 \geq 0$	$\binom{n-1}{2}$
$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$			
k summands	$n = x_1 + \cdots + x_k$	$x_1, \ldots, x_k \geq 1$	$n-k = y_1 + \cdots + y_k$	$y_1, \ldots, y_k \geq 0$	$\binom{n-1}{k-1}$
$\cdots\cdots$	$\cdots\cdots$	$\cdots\cdots$			
n summands	$n = x_1 + \cdots + x_n$	$x_1, \ldots, x_n \geq 1$	$n-n = y_1 + \cdots + y_n$	$y_1, \ldots, y_n \geq 1$	$\binom{n-1}{n-1}$

The total yeilds the answer:

$$1 + \binom{n-1}{1} + \cdots + \binom{n-1}{k-1} + \cdots + \binom{n-1}{n-1} = (1+1)^{n-1} = 2^{n-1}$$

(10) Consider the following segment of a code:

```
for i = 1 to 2019
    for j = 1 to i
        for k = 1 to j
            print(i + j + k)
```

Here the variables i, j, k are integers. How many times the command $\texttt{print}(i+j+k)$ will be executed if $1 \le k \le j \le i \le 2019$? In fact, we have to count how many triples of integers (i, j, k) satisfies the condition:

$$1 \le k \le j \le i \le 2019.$$

To answer the question, we imagine 2019 empty boxes. Then any placement of 3 objects into those 2019 boxes counts exactly one execution. The answer is

$$\binom{3+2019-1}{2019-1} = \binom{2021}{2018} = \binom{2021}{3} = \frac{2021 \cdot 2020 \cdot 2019}{1 \cdot 2 \cdot 3}$$

(11) How many times the command $\texttt{print}(i+j+k+\ell)$ will be executed in the following segment of a code if $1 \le k \le j \le i \le 2019$?

```
for i = 1 to n
    for j = 1 to i
        for k = 1 to j
            for ℓ = 1 to k
                print(i + j + k + ℓ)
```

(12) The Catalan numbers. Let us consider the xy-plane, and two types of moves:

$$\mathsf{R} : (x, y) \mapsto (x+1, y), \quad \mathsf{U} : (x, y) \mapsto (x, y+1).$$

We are allowed to make the moves R and U to get from the point $(0, 0)$ to the point (n, n). A path consisting of only the moves R and U is called **monotonic**.

Warm-up question: How many monotonic paths are there from $(0, 0)$ to (n, n)?

This is easy. Indeed, any monotonic path can be recorded as a sequence of n R's and n U's. A total number of moves is $2n$; thus it is enough to choose n slots for R's (or n U's). We obtain $\binom{2n}{n}$ paths.

A monotonic path from $(0, 0)$ to (n, n) is **dangerous** if it crosses the diagonal.

Actual question: How many non-dangerous monotonic paths are there from $(0, 0)$ to (n, n)?

Let $n = 6$. Then the paths

R R U R U U R U R U R U   is non-dangerous,

R R U R U U R U U U R R   is dangerous.

To distinguish dangerous and non-dangerous paths, we count how many R and U moves did we make at every step:

10 20 21 31 32 33 43 44 54 55 65 66
R R U R U U R U R U R U   is non-dangerous,

⇓
10 20 21 31 32 33 43 44 45 46 56 56
R R U R U U R U U U R R   is dangerous.

Moreover, once the number of U-moves gets greater than the number of R-moves, we use the red color. Then, once the first red indicator appears, we write new path, where we change the path after the dangerous U-move: all R-moves we turn to U-moves, and all U-moves we turn to R-moves:

⇓
10 20 21 31 32 33 43 44 45 46 56 56
R R U R U U R U U U R R   a dangerous path.

⇓
10 20 21 31 32 33 43 44 45
R R U R U U R U U R U U   new path.

In the black portion of the new path, we have 4 R-moves and 5 U-moves; in the red portion, we have 1 R-move and 2 U-moves. Totally, new path has 5 R-moves and 7 U-moves. Thus it is a path from $(0,0)$ to $(5,7)$. We claim that in this way every dangerous path turns to a path from $(0,0)$ to $(5,7)$. Thus we have the answer:

$$\{\#\text{ of all paths}\} - \{\#\text{ of dangerous paths}\} = \binom{12}{6} - \binom{12}{5}.$$

For general n, we do the same. Namely, we consider a dangerous path (first line) and we produce new path below:

	⇓	
$(k-1)$ U's, $(k-1)$ R's	U	$(n-k)$ U's, $(n-k+1)$ R's
$(k-1)$ U's, $(k-1)$ R's	U	$(n-k)$ R's, $(n-k+1)$ U's

The first path is dangerous since the red marker ⇓ shows that there are k U's and $(k-1)$ R's, so the path crossed the diagonal. For the new path we changed all U's by R's and all R's by U's **after** the red marker ⇓. Totally, for the new path, we have

$$\begin{aligned} k+n-k+1 &= n+1 \quad \mathsf{U}'s \\ k-1+n-k &= n-1 \quad \mathsf{R}'s \end{aligned}$$

Thus we have the answer:

$$b_n := \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$