Summary on Lecture 1, January 7, 2019

1. Warm-up examples.

Notations: $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$ integers, $\mathbf{Z}_+ := \{1, 2, 3, \ldots\}$ natural numbers.
The problems discussed:

(1) Let $n \in \mathbf{Z}_+$. There are n integers i such that $1 \le i \le n$.

(2) Let $m, n \in \mathbf{Z}_+$, $m \le n$. Then there are $n - m + 1$ integers i such that $m \le i \le n$.

(3) Let $\ell, n \in \mathbf{Z}_+$. How many integers i such that $i = \ell \cdot j$, and $1 \le i \le n$? The answer: $\lfloor \frac{n}{\ell} \rfloor$.

(4) Let $\ell, m, n \in \mathbf{Z}_+$. How many integers i such that $i = \ell \cdot j$, and $m \le i \le n$? The answer: $\lfloor \frac{n}{\ell} \rfloor - \lfloor \frac{m-1}{\ell} \rfloor$.

(5) Let $S = \{1, 2, \ldots, 100,000\}$, and $p \le n$. Let

$$A_p = \{ \ i \in S \mid i = p \cdot j \ \text{ for some } j \in \mathbf{Z}_+ \ \}.$$

How many intergers i in S are such that $i \in A_7$ or $i \in A_{11}$? The answer:

$$\begin{array}{rl} & \lfloor \frac{100,000}{7} \rfloor + \lfloor \frac{100,000}{11} \rfloor - \lfloor \frac{100,000}{77} \rfloor \\ = & 14,285 + 9,090 - 1,298 \\ = & 22,077 \end{array}$$

(5′) Let $S = \{1, 2, \ldots, 100,000\}$, and $p \le n$. How many intergers i in S which are divisible by 7 or by 11, but not by both? The answer:

$$\begin{array}{rl} & \lfloor \frac{100,000}{7} \rfloor + \lfloor \frac{100,000}{11} \rfloor - \lfloor \frac{100,000}{77} \rfloor \\ = & 14,285 + 9,090 - 2 \cdot 1,298 \\ = & 20,779 \end{array}$$

2. Union (sum) Rule. Let A, B be finite sets. Then $|A \cup B| = |A| + |B|$ if $A \cap B = \emptyset$. In general,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

3. Product Rule. Suppose that a set of ordered k-tuples $(s_1, s_2, \ldots, s_k)$ has the following structure. There are n_1 possible choices of s_1. Given an s_1, there are n_2 possible choices of s_2; given any s_1 and s_2, there are n_3 possible choices of s_3; and in general, given any s_1, $s_2, \ldots, s_{j-1}$, there are n_j choices of s_j. Then the set has $n_1 n_2 \ldots n_k$ elements. In particular, for finite sets S_1, $S_2, \ldots S_k$ we have $|S_1 \times S_2 \times \cdots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k|$.

(6) How many 2-digit integers are there? The answer: $9 \cdot 10$.

(7) How many odd 2-digit integers are there? The answer: $9 \cdot 5$.

(8) A license plate has first 3 letters and then 3 digits.

   (a) How many license plates are there? The answer: $26^3 \cdot 10^3$.

(b) Assume that no letters are repeated. How many license plates are there? The answer: $26 \cdot 25 \cdot 24 \cdot 10^3$.

(c) Assume that no letters and no digits are repeated. How many license plates are there? The answer: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$.

(9) Let $\Sigma = \{a_1, \dots, a_n\}$ be an alphabet.

(a) How many words of length k are there? The answer: n^k.

(b) How many words without repetition of length k are there? The answer: $P(n,k) := n(n-1)\cdots(n-k+1)$.

4. Permutations. Let $\ell! := \ell \cdot (\ell-1)\cdots 3 \cdot 2 \cdot 1$. Convention: $0! = 1$. Then $P(n,k) = \frac{n!}{(n-k)!}$. More general problem: There are n objects with n_i indistinguishable objects of i-th type, with $i = 1, \dots, s$, and $n = n_1 + \cdots + n_s$. Then there are

$$\begin{pmatrix} & n & \\ n_1 & \cdots & n_s \end{pmatrix} = \frac{n!}{n_1! \cdots n_s!}$$

linear arrangements of such n objects.

(10) How many words are there which are given by permutation of letters in the word $BALL$? Hint: BAL_1L_2. The answer: $\frac{4!}{2!}$.

(11) How many words are there which are given by permutation of letters in the word $DATABASES$? Hint: $DA_1TA_2BA_3S_1ES_2$. The answer: $\frac{9!}{3! \cdot 2!}$.

(12) How many words are there which are given by permutation of letters in the word $SOCIOLOGICAL$? Hint: $SO_1C_1IO_2L_1O_3GIC_2AL_2$. The answer: $\frac{12!}{3! \cdot 2! \cdot 2!}$.

(13) How many words are there which are given by permutation of letters in the word $MASSASAUGA$? Hint: $MA_1S_1S_2A_2SA_3UGA_4$?. The answer: $\frac{10!}{4! \cdot 3!}$.

(14) How many words are there which are given by permutation of letters in the word $MASSASAUGA$, so that all letters A are placed together? Hint: consider the symbols $AAAA$, M, S_1, S_2, S_3, U, G. The answer: $\frac{7!}{3!}$.

(15) Show that $\frac{(2k)!}{2^k}$ is an integer. Hint: consider the symbols $a_1, a_1, a_2, a_2, \dots, a_k, a_k$.

(16) How many monotonic paths are there from $(0,0)$ to (n,k)? Here a path is monotonic, if only moves allowed are steps right and steps up. Hint: let R stand for a single move to the right, and U stand for a single move up. Then any monotonic path is given as a word with n letters R and k letters U. The answer: $\frac{(n+k)!}{n! \cdot k!}$.

(17) There are 8 people A, B, C, D, E, F, G, H to be seated about round table. (Two seatings arrangements are the same up to a rotation.) How many seatings arrangements are possible? The answer: $\frac{8!}{8}$.

(18) There are 8 people A, B, C, D, E, F, G, H to be seated about round table. Assume, in addition, that A, B, C, D are Females and E, F, G, H are Males. We consider only such seatings arrangements that no two male or females are seating next to each other. How many seatings such arrangements are there? The answer: $\frac{1}{2} \cdot \frac{4!}{4} \cdot \frac{4!}{4}$.